Find The Dimensions Of The Closed Right Circular Cylindrical Can Of Smallest Surface Area Whose Volume

Find The Dimensions Of The Closed Right Circular Cylindrical Can Of Smallest Surface Area Whose Volume is a classic optimization problem in calculus and geometric design. It involves determining the dimensions—specifically, the radius and height—of a cylindrical can that minimizes material usage while maintaining a specified volume. This problem is fundamental in manufacturing, packaging, and engineering, where efficiency and cost-effectiveness are crucial.

In this comprehensive article, we will explore the problem step-by-step, covering the mathematical formulation, derivation of the conditions for optimality, and practical considerations. By the end, you'll understand how to find the optimal dimensions to minimize surface area for a given volume, along with the underlying calculus principles.

Understanding the Problem

The problem involves a right circular cylindrical can that is closed at both ends. The goal is to find the dimensions—radius (r) and height (h)—that result in the smallest possible surface area, given a fixed volume (V).

Key Components:

- Shape: Closed right circular cylinder
- Variables: Radius (r) and height (h)
- Objective: Minimize surface area (S)
- Constraint: Fixed volume (V)

This type of problem involves constrained optimization, where the objective function (surface area) is minimized under a specific constraint (volume).

Mathematical Formulation

To approach the problem mathematically, we need to express the relevant quantities:

1. Volume of a Cylinder

The volume (V) of a cylinder is given by:

$$ V = \pi r^2 h $$

Since volume is fixed, we can express one variable in terms of the other:

$$ h = \frac{V}{\pi r^2} $$

2. Surface Area of a Cylinder

The surface area (S) includes the lateral surface and the area of the two circular ends:

$$ S = 2\pi r h + 2\pi r^2 $$

Substituting the expression for h:

$$ S(r) = 2\pi r \times \frac{V}{\pi r^2} + 2\pi r^2 $$

Simplify the expression:

$$ S(r) = 2V \times \frac{1}{r} + 2 \pi r^2 $$

Thus, the problem reduces to finding the radius r that minimizes:

$$ S(r) = \frac{2V}{r} + 2 \pi r^2 $$

with $r > 0$.

Optimization Process

To find the optimal radius r, we differentiate the surface area function with respect to r and set the derivative equal to zero to find critical points.

1. Derivative of Surface Area Function

Calculate $S'(r)$:

$$ S'(r) = - \frac{2V}{r^2} + 4 \pi r $$

2. Critical Points

Set $S'(r) = 0$:

$$ - \frac{2V}{r^2} + 4 \pi r = 0 $$

Rearranged:

$$ 4 \pi r = \frac{2V}{r^2} $$

Multiply both sides by r^2:

$$ 4 \pi r^3 = 2V $$

Solve for r:

$$ r^3 = \frac{2V}{4 \pi} = \frac{V}{2 \pi} $$

$$ r = \left(\frac{V}{2 \pi} \right)^{1/3} $$

3. Find Height h

Recall the expression for height:

$$ h = \frac{V}{\pi r^2} $$

Substitute the optimal r:

$$ h = \frac{V}{\pi \left(\frac{V}{2 \pi} \right)^{2/3}} $$

Simplify numerator and denominator:

$$ h = \frac{V}{\pi} \times \left(\frac{2 \pi}{V} \right)^{2/3} $$

Expressed more cleanly, the height becomes:

$$ h = \left(\frac{V}{\pi} \right) \times \left(\frac{2 \pi}{V} \right)^{2/3} $$

Alternatively, for clarity, you can write:

$$ h = \left(\frac{V}{\pi} \right)^{1/3} \times \left(2 \pi \right)^{2/3} $$

Summary of the optimal dimensions:

- Optimal radius:

$$ r_{\text{opt}} = \left(\frac{V}{2 \pi} \right)^{1/3} $$

- Optimal height:

$$ h_{\text{opt}} = \frac{V}{\pi r_{\text{opt}}^2} $$

which simplifies to:

$$ h_{\text{opt}} = 2 r_{\text{opt}} $$

This indicates that, for the minimal surface area, the height of the cylinder should be twice the radius.

Interpretation of Results

The derived relationship, $h = 2r$, reveals a key insight: the most material-efficient cylindrical can for a given volume is one where the height is twice the radius.

This result aligns with practical observations in manufacturing, where cylinders with this proportion tend to use materials efficiently. The symmetry and balance between height and radius minimize the surface area, reducing costs and material waste.

Practical Example

Suppose you need to design a cylindrical can with a volume of 1000 cubic centimeters (cm^3). Let's compute the optimal dimensions.

Step 1: Calculate the optimal radius:

$$ r = \left(\frac{V}{2 \pi} \right)^{1/3} = \left(\frac{1000}{2 \pi} \right)^{1/3} $$

Numerical calculation:

$$ \frac{1000}{2 \pi} \approx \frac{1000}{6.2832} \approx 159.15 $$

$$ r \approx (159.15)^{1/3} \approx 5.36 \text{ cm} $$

Step 2: Calculate the height:

$$ h = 2r \approx 2 \times 5.36 \approx 10.72 \text{ cm} $$

Step 3: Verify the volume:

$$ V = \pi r^2 h \approx 3.1416 \times (5.36)^2 \times 10.72 $$

$$ V \approx 3.1416 \times 28.75 \times 10.72 \approx 3.1416 \times 308.36 \approx 968.66 \text{ cm}^3 $$

Close to 1000 cm^3, with minor differences due to rounding. Fine-tuning the dimensions would yield even closer results.

Additional Considerations

While the mathematical derivation provides the ideal dimensions for minimizing surface area, real-world factors might influence the final design.

1. Manufacturing Constraints

- Material thickness: Real cans have material thickness, which affects total surface area and weight.

- Standard sizes: Manufacturing processes may prefer standard sizes, slightly deviating from the ideal.
- Structural integrity: Taller or thinner cans might be more prone to deformation.

2. Cost and Material Savings

Minimizing surface area directly translates to less material used, reducing costs. However, other factors like ease of filling, stacking, and transportation may influence the dimensions.

3. Generalization to Other Shapes

The approach used here can be extended to other shapes, such as cones or rectangular prisms, for similar optimization problems.

Conclusion

Finding the dimensions of a closed right circular cylindrical can with the smallest surface area for a fixed volume involves setting up the problem as a constrained optimization task. By expressing the surface area in terms of one variable, differentiating, and solving for critical points, we discover that the optimal configuration occurs when the height is twice the radius.

This problem exemplifies the power of calculus in solving real-world engineering and manufacturing challenges. Whether designing packaging, containers, or structural elements, understanding how to optimize dimensions leads to more efficient and cost-effective solutions.

Key takeaways:

- The optimal radius $r = \left(\frac{V}{2 \pi} \right)^{1/3}$
- The optimal height $h = 2r$
- The minimized surface area can be computed using these dimensions

By applying these principles, engineers and designers can create cylinders that are material-efficient, cost-effective, and structurally sound.

References:

- Stewart, J. (2015). Calculus: Early Transcendentals. Cengage Learning.
- Anton, H., Bivens, I., & Davis, S. (2013). Calculus: Early Transcendental. John Wiley & Sons.
- Mathematics for Engineers and Scientists, OpenStax.

Frequently Asked Questions

What is the main goal when finding the dimensions of the smallest surface area closed right circular cylindrical can with a given volume?

The main goal is to determine the height and radius that minimize the surface area while maintaining a fixed volume for the can.

How do you set up the problem of minimizing surface area for a cylindrical can with a given volume?

You express the surface area and volume in terms of the radius and height, then use calculus techniques such as differentiation and optimization to find the dimensions that minimize surface area.

What formulas are essential for solving this problem?

The key formulas are the volume of a cylinder, $V = \pi r^2 h$, and the surface area, $A = 2\pi r^2 + 2\pi rh$, where r is the radius and h is the height.

Why is the derivative used in finding the minimal surface area in this problem?

The derivative helps identify critical points where the surface area function reaches a minimum by setting the derivative equal to zero and solving for the dimensions.

What is the optimal relationship between the radius and height of the cylinder for minimum surface area?

The optimal solution often results in the height being twice the radius, i.e., $h = 2r$, which balances the contributions of lateral surface area and top/bottom area.

Can you explain the significance of the Lagrange multipliers method in this context?

Lagrange multipliers provide a systematic way to optimize the surface area while satisfying the volume constraint, especially when multiple variables are involved.

How do you verify that the obtained dimensions indeed give the minimum surface area?

You check the second derivative or analyze the behavior of the function around the critical points to confirm that the solution corresponds to a minimum rather than a maximum or saddle point.

What practical applications use the concept of minimizing surface area for cylindrical cans?

This concept is used in manufacturing to reduce material costs, optimize packaging, and design containers that are efficient in terms of material usage while maintaining volume requirements.

Additional Resources

Find The Dimensions Of The Closed Right Circular Cylindrical Can Of Smallest Surface Area Whose Volume

When it comes to optimizing container designs, especially for storage, packaging, and manufacturing, the problem of determining the dimensions of a closed right circular cylindrical can with minimal surface area for a given volume is both classical and highly relevant. This problem combines principles of calculus, geometry, and optimization to identify the most efficient shape that minimizes material use while satisfying volume constraints. In this comprehensive review, we will delve into the mathematical formulation, solution techniques, and practical implications of this optimization problem.

Understanding the Problem

The core objective is to find the dimensions—specifically, the radius (r) and height (h)—of a closed right circular cylindrical can such that:

- The volume (V) is fixed at a specified value.
- The surface area (S) is minimized.

This problem is an example of a classical calculus of variations and constrained optimization problem, often encountered in engineering design, manufacturing, and material science.

Mathematical Formulation

Variables and Parameters

- Let r denote the radius of the circular base of the cylinder.
- Let h denote the height of the cylinder.
- The volume (V) of the cylinder is given by:

$$
V = \pi r^2 h
$$

- The surface area (S) of the closed cylinder includes the lateral surface and the two circular bases:

$$
S = 2\pi r h + 2 \pi r^2
$$

Objective and Constraints

- Objective: Minimize $S(r,h) = 2 \pi r h + 2 \pi r^2$
- Constraint: $V = \pi r^2 h = V_0$ (a known constant)

The problem reduces to minimizing $S(r,h)$ subject to the volume constraint $V = V_0$.

Mathematical Approach: Using Calculus and Lagrange Multipliers

The problem involves a constrained optimization which can be tackled efficiently using Lagrange multipliers.

Setting up the Lagrangian

Define the Lagrangian function:

$$
\mathcal{L}(r,h,\lambda) = 2 \pi r h + 2 \pi r^2 + \lambda (\pi r^2 h - V_0)
$$

where λ is the Lagrange multiplier associated with the volume constraint.

Partial Derivatives and Conditions for Optimality

To find the stationary points, take the partial derivatives:

1. Differentiating with respect to r:

$$
\frac{\partial \mathcal{L}}{\partial r} = 2 \pi h + 4 \pi r + \lambda (2 \pi r h) = 0
$$

2. Differentiating with respect to h:

\[

\frac{\partial \mathcal{L}}{\partial h} = 2 \pi r + \lambda (\pi r^2) = 0

\]

3. The constraint itself:

\[

\pi r^2 h = V_0

\]

Solution Derivation

From the second condition:

\[

2 \pi r + \lambda \pi r^2 = 0 \implies 2 r + \lambda r^2 = 0

\]

Assuming \(r \neq 0 \),

\[

2 + \lambda r = 0 \implies \lambda = -\frac{2}{r}

\]

Substituting into the first condition:

\[

2 \pi h + 4 \pi r + \left(-\frac{2}{r}\right) (2 \pi r h) = 0

\]

Simplify:

$$
2 \pi h + 4 \pi r - 4 \pi h = 0
$$

$$
(2 \pi h - 4 \pi h) + 4 \pi r = 0
$$

$$
-2 \pi h + 4 \pi r = 0
$$

Divide through by 2π:

$$
- h + 2 r = 0 \implies h = 2 r
$$

Now, use the volume constraint:

$$
V_0 = \pi r^2 h = \pi r^2 (2 r) = 2 \pi r^3
$$

Solve for r:

$$
r^3 = \frac{V_0}{2 \pi} \implies r = \left(\frac{V_0}{2 \pi}\right)^{1/3}
$$

And the height:

$$
h = 2 r = 2 \left(\frac{V_0}{2 \pi}\right)^{1/3}
$$

Optimal Dimensions of the Cylinder

Final expressions for the dimensions:

$$
\boxed{
r_{opt} = \left(\frac{V_0}{2 \pi}\right)^{1/3}
}
$$

$$
\boxed{
h_{opt} = 2 \left(\frac{V_0}{2 \pi}\right)^{1/3}
}
$$

The implications are clear: the height of the minimally surfaced cylinder is twice its radius for a fixed volume.

Physical and Practical Interpretation

The derived relationship suggests that, for the most material-efficient design of a closed cylindrical can at a fixed volume:

- The height should be twice the radius.
- The optimal dimensions depend on the cube root of the volume, scaled by constants related to π.

This result aligns with intuition and classical findings in minimal surface problems, where the shape tends towards a proportion that balances surface area and volume constraints.

Calculating the Surface Area for Optimal Dimensions

Once the optimal dimensions are known, the minimal surface area can be computed:

$$
S_{min} = 2 \pi r h + 2 \pi r^2
$$

Substitute $h = 2 r$:

$$
S_{min} = 2 \pi r (2 r) + 2 \pi r^2 = 4 \pi r^2 + 2 \pi r^2 = 6 \pi r^2
$$

Express r in terms of V_0:

$$
r = \left(\frac{V_0}{2 \pi}\right)^{1/3}
$$

So,

$$
S_{min} = 6 \pi \left(\frac{V_0}{2 \pi}\right)^{2/3}
$$

This formula allows for direct calculation of minimal surface area based on the volume.

Special Cases and Numerical Examples

To solidify understanding, consider a specific volume:

Example: Suppose $V_0 = 1000\, \text{cm}^3$.

1. Compute r:

$$
r = \left(\frac{1000}{2 \pi}\right)^{1/3} \approx \left(\frac{1000}{6.2832}\right)^{1/3} \approx (159.15)^{1/3} \approx 5.36\, \text{cm}
$$

2. Compute h:

$$

$$
h = 2 r \approx 2 \times 5.36 \approx 10.72\, \text{cm}
$$

3. Compute minimal surface area:

$$
S_{min} = 6 \pi r^2 \approx 6 \times 3.1416 \times (5.36)^2 \approx 6 \times 3.1416 \times 28.73 \approx 6 \times 90.22 \approx 541.3\, \text{cm}^2
$$

This practical example demonstrates how the theoretical formulas translate into real-world measurements.

Additional Considerations and Real-World Constraints

While the mathematical solution provides ideal dimensions, real-world applications often impose additional constraints:

- Manufacturing Limitations: Minimum or maximum feasible dimensions.
- Material Strength: Thinner walls may compromise durability.
- Aesthetic or Functional Requirements: Height or radius preferences.
- Cost Factors: Material costs may influence design choices beyond pure surface area considerations.

Understanding these factors helps engineers and designers tailor the theoretical optimal dimensions to practical, manufacturable solutions.

Extensions and Related Problems

The problem of minimizing surface area for fixed volume extends beyond cylinders to other shapes:

- Sphere: The shape with the smallest surface area for a given volume.
- Cubes: For rectangular boxes, the cube minimizes surface area.
- Other Geometries: Optimization of composite shapes or shapes with constraints.

In addition, similar optimization strategies are applicable in:

- Designing storage tanks
- Manufacturing packaging
- Material science applications

Conclusion

The mathematical investigation into the dimensions of a closed right circular cylindrical can of smallest surface area for a fixed volume reveals elegant and practical results:

- The optimal ratio of height to

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