Find The General Solution By Solving The Differential Equation Y'' + 4y' + 4y = Cos X Using The Method

Find The General Solution By Solving The Differential Equation Y'' + 4y' + 4y = Cos X Using The Method

When working with differential equations, finding the general solution is a crucial step in understanding the behavior of the system described by the equation. In this article, we will explore how to find the general solution of the differential equation:

$$
Y'' + 4Y' + 4Y = \cos X
$$

using the method of solving linear nonhomogeneous differential equations. This involves two main parts: solving the corresponding homogeneous equation and finding a particular solution to the nonhomogeneous part. By systematically applying these methods, we can derive the complete general solution.

Understanding the Differential Equation

The differential equation in question:

$$
Y'' + 4Y' + 4Y = \cos X
$$

is a second-order linear differential equation with constant coefficients. The left side represents a homogeneous differential operator, while the right side, $\cos X$, is a nonhomogeneous term (forcing function).

Key components:
- Y'': the second derivative of Y
- Y': the first derivative
- Y: the original function
- $\cos X$: the nonhomogeneous (forcing) function

Step-by-Step Solution Approach

To find the general solution, we follow these main steps:

1. Find the complementary (homogeneous) solution Y_h by solving the homogeneous equation:

   $$
   Y'' + 4Y' + 4Y = 0
   $$

2. Find a particular solution Y_p to the nonhomogeneous equation:

   $$
   Y'' + 4Y' + 4Y = \cos X
   $$

3. Combine the homogeneous and particular solutions to get the general solution:

   $$
   Y = Y_h + Y_p
   $$

Solving the Homogeneous Equation

The homogeneous equation:

$$
Y'' + 4Y' + 4Y = 0
$$

is a linear differential equation with constant coefficients. To solve it, we look for solutions of the form:

$$
Y_h = e^{rx}
$$

where r is a constant to be determined.

Characteristic Equation:

$$
r^2 + 4r + 4 = 0
$$

\]

This quadratic can be factored or solved using the quadratic formula:

$$
r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
$$

with $a=1$, $b=4$, and $c=4$.

Calculating the discriminant:

$$
\Delta = 4^2 - 4 \times 1 \times 4 = 16 - 16 = 0
$$

Since the discriminant is zero, the roots are real and repeated:

$$
r = \frac{-4 \pm 0}{2} = -2
$$

Therefore, the characteristic roots are:

$$
r = -2 \text{ (double root)}
$$

Homogeneous Solution:

For a repeated root r, the general solution is:

$$
Y_h = (A + Bx) e^{rx} = (A + Bx) e^{-2x}
$$

where A and B are arbitrary constants.

Finding a Particular Solution Y_p

Since the right side of the differential equation is $\cos X$, which is a nonhomogeneous term, we need to find a particular solution.

Method: The Method of Undetermined Coefficients

Given the form of the nonhomogeneous term, we choose a trial particular solution of the form:

$$
Y_p = M \cos X + N \sin X
$$

where M and N are coefficients to be determined.

Note: Because the homogeneous solution involves exponential functions, and the right side involves trigonometric functions, the chosen trial form is appropriate and does not overlap with the homogeneous solution.

Calculating Derivatives of Y_p

$$
Y_p = M \cos X + N \sin X
$$

$$
Y_p' = -M \sin X + N \cos X
$$

$$
Y_p'' = -M \cos X - N \sin X
$$

Substituting into the Differential Equation

Substitute Y_p, Y_p', and Y_p'' into the original equation:

$$
Y'' + 4Y' + 4Y = \cos X
$$

which becomes:

$$
(-M \cos X - N \sin X) + 4(-M \sin X + N \cos X) + 4(M \cos X + N \sin X) = \cos X
$$

Group like terms:

$$
\begin{aligned}
& (-M \cos X - N \sin X) \\

& + 4(-M \sin X + N \cos X) \\
& + 4(M \cos X + N \sin X) \\
= \left[-M \cos X - N \sin X \right] + \left[-4M \sin X + 4N \cos X \right] + \left[4M \cos X + 4N \sin X \right]
\end{aligned}
\]

Combine coefficients for \(\cos X\) and \(\sin X\):

\[
\text{Coefficient of } \cos X:
\]
\[
(-M) + 4N + 4M = (-M + 4M) + 4N = 3M + 4N
\]

\[
\text{Coefficient of } \sin X:
\]
\[
(-N) - 4M + 4N = (-N + 4N) - 4M = 3N - 4M
\]

So, the entire left side simplifies to:

\[
(3M + 4N) \cos X + (3N - 4M) \sin X
\]

Set this equal to the right side \(\cos X\):

\[
(3M + 4N) \cos X + (3N - 4M) \sin X = \cos X
\]

Matching coefficients:

\[
3M + 4N = 1 \quad \text{(coefficient of } \cos X)
\]

\[
3N - 4M = 0 \quad \text{(coefficient of } \sin X)
\]

Solving for \(M\) and \(N\)

From the second equation:

$$
3N = 4M \Rightarrow N = \frac{4M}{3}
$$

Substitute into the first:

$$
3M + 4 \times \frac{4M}{3} = 1
$$

$$
3M + \frac{16M}{3} = 1
$$

Multiply through by 3 to clear denominator:

$$
9M + 16M = 3
$$

$$
25M = 3
$$

$$
M = \frac{3}{25}
$$

Now, find N:

$$
N = \frac{4 \times \frac{3}{25}}{3} = \frac{\frac{12}{25}}{3} = \frac{12}{25} \times \frac{1}{3} = \frac{12}{75} = \frac{4}{25}
$$

Particular Solution:

$$
Y_p = \frac{3}{25} \cos X + \frac{4}{25} \sin X
$$

Constructing the General Solution

The general solution is the sum of the homogeneous and particular solutions:

$$
Y = Y_h + Y_p
$$

$$
Y = (A + Bx) e^{-2x} + \frac{3}{25} \cos X + \frac{4}{25} \sin X
$$

where A, B are arbitrary constants determined by initial conditions if provided.

Summary of the Solution Process

To summarize, solving the differential equation $Y'' + 4Y' + 4Y = \cos X$ involves:

- Finding the characteristic roots of the homogeneous equation, which are repeated roots at -2.
- Constructing the homogeneous solution based on these roots: $(A + Bx) e^{-2x}$.
- Choosing an appropriate trial for the particular solution based on the nonhomogeneous term—here, trigonometric functions—and calculating the coefficients by substitution.
- Combining both solutions to obtain the general solution.

Additional Tips for Solving Similar Differential Equations

- Always check the form of the nonhomogeneous term to pick an appropriate

Frequently Asked Questions

What is the general approach to solving the differential equation Y'' + 4Y' + 4Y = Cos X using the method of solving differential equations?

The general approach involves finding the complementary (homogeneous) solution by solving Y'' + 4Y' + 4Y = 0, and then finding a particular solution using methods like undetermined coefficients or variation of parameters for the non-homogeneous part Cos X. The sum of these solutions gives the general solution.

How do you find the complementary solution for Y'' + 4Y' + 4Y = Cos X?

First, solve the homogeneous equation Y'' + 4Y' + 4Y = 0 by finding the roots of the characteristic equation r^2 + 4r + 4 = 0, which are r = -2 (double root). The complementary solution is then Y_c = (A + Bx) e^{-2x}.

Which method should be used to find the particular solution for Y'' + 4Y' + 4Y = Cos X?

Since the non-homogeneous term is Cos X, the method of undetermined coefficients is suitable. Assume a particular solution of the form Y_p = C cos X + D sin X and determine C and D by substituting into the differential equation.

How do you determine the coefficients C and D for the particular solution?

Substitute Y_p = C cos X + D sin X into the differential equation, compute Y'_p and Y''_p, and then equate coefficients of cos X and sin X on both sides to solve for C and D.

What is the final form of the general solution after solving the differential equation Y'' + 4Y' + 4Y = Cos

X?

The general solution is $Y = (A + Bx) e^{-2x} + C \cos X + D \sin X$, where A, B, C, and D are arbitrary constants found through initial conditions or boundary conditions.

Why does the particular solution for this differential equation include both cosine and sine terms?

Because the non-homogeneous term is Cos X, and the differential equation involves derivatives that can produce both cosine and sine functions, the particular solution naturally includes both to account for the right-hand side's form during the method of undetermined coefficients.

Additional Resources

Find The General Solution By Solving The Differential Equation $Y'' + 4y' + 4y = \cos X$ Using The Method

---

Introduction

Differential equations are fundamental tools in mathematical modeling, physics, engineering, and many scientific disciplines. They describe the relationship between a function and its derivatives, often representing dynamic systems' behavior. The second-order linear differential equation with constant coefficients, such as

$$ Y'' + 4Y' + 4Y = \cos X, $$

is a standard form encountered in many applications, including oscillatory systems, electrical circuits, and mechanical vibrations.

In this detailed review, we will explore how to find the general solution of the differential equation $Y'' + 4Y' + 4Y = \cos X$ using the classical method of solving linear differential equations with constant coefficients. We will dissect each step, justify the methods used, and explain the underlying concepts to provide a comprehensive understanding.

The Structure of the Differential Equation

Before diving into the solution process, it is crucial to analyze the structure of the differential equation:

$$ Y'' + 4Y' + 4Y = \cos X. $$

This is a second-order linear nonhomogeneous differential equation with constant coefficients, where:

- Y'' is the second derivative of the unknown function Y with respect to X.
- Y' is the first derivative.
- The right-hand side $\cos X$ is a nonhomogeneous term (forcing function).

The solution process involves two main parts:

- Finding the complementary (homogeneous) solution Y_h, which solves the associated homogeneous equation.
- Finding a particular solution Y_p, which satisfies the entire nonhomogeneous equation.

The general solution is then:

$$ Y = Y_h + Y_p. $$

Step 1: Solving the Homogeneous Equation

Formulating the Homogeneous Equation

The associated homogeneous differential equation is:

$$ Y'' + 4Y' + 4Y = 0. $$

This is a standard linear differential equation with constant coefficients.

Characteristic Equation

To find the complementary solution, we derive the characteristic equation:

$$ r^2 + 4r + 4 = 0. $$

This quadratic equation in r captures the behavior of the solutions.

Solving the Characteristic Equation

The quadratic factors as:

$$ (r + 2)^2 = 0, $$

which yields a repeated root:

$$ r = -2. $$

Homogeneous Solution

Given the repeated root $r = -2$, the general form of the homogeneous solution is:

$$Y_h = (A + Bx) e^{-2x},$$

where A and B are arbitrary constants determined by initial conditions.

Summary of homogeneous solution:

$$\boxed{
Y_h = (A + Bx) e^{-2x}
}$$

Step 2: Finding a Particular Solution

Since the right-hand side is $\cos X$, a typical approach is to assume a particular solution involving sinusoidal functions.

Methodology for Particular Solution

- Method of Undetermined Coefficients: Suitable when the nonhomogeneous term is a simple function like sine or cosine.
- Since $\cos X$ is on the right, we try assuming a particular solution of the form:

$$Y_p = P \cos X + Q \sin X,$$

where P and Q are constants to be determined.

Step 3: Computing Derivatives of the Particular Solution

Given:

$$Y_p = P \cos X + Q \sin X,$$

we compute:

- First derivative:

$$Y_p' = -P \sin X + Q \cos X,$$

- Second derivative:

$$Y_p'' = -P \cos X - Q \sin X.$$

Step 4: Substituting into the Differential Equation

Plugging Y_p, Y_p', and Y_p'' into the original equation:

$$ Y'' + 4Y' + 4Y = \cos X, $$

we get:

$$
(-P \cos X - Q \sin X) + 4(-P \sin X + Q \cos X) + 4(P \cos X + Q \sin X) = \cos X.
$$

Simplify term-by-term:

$$
\begin{aligned}
& -P \cos X - Q \sin X \\
& - 4P \sin X + 4Q \cos X \\
& + 4P \cos X + 4Q \sin X
\end{aligned}
$$

Group similar terms:

- Coefficients of $\cos X$:

$$
(-P) + (4Q) + (4P) = (-P + 4P) + 4Q = 3P + 4Q,
$$

- Coefficients of $\sin X$:

$$
(-Q) - 4P + 4Q = (-Q + 4Q) - 4P = 3Q - 4P.
$$

The overall equation simplifies to:

$$
(3P + 4Q) \cos X + (3Q - 4P) \sin X = \cos X.
$$

Step 5: Equating Coefficients

Since the right-hand side is $\cos X$, which can be written as:

$$ 1 \cdot \cos X + 0 \cdot \sin X, $$

we match coefficients:

\[

\boxed{
\begin{cases}
3P + 4Q = 1, \\
3Q - 4P = 0.
\end{cases}
}
\]

This system of equations allows us to solve for \(P \) and \(Q \).

Step 6: Solving for \(P \) and \(Q \)

From the second equation:

\[3Q = 4P \Rightarrow Q = \frac{4}{3} P. \]

Substitute into the first:

\[3P + 4 \left(\frac{4}{3} P \right) = 1, \]

which simplifies to:

\[3P + \frac{16}{3} P = 1. \]

Multiply through by 3 to clear denominators:

\[9P + 16P = 3, \]

\[25P = 3, \]

\[P = \frac{3}{25}. \]

Now, find \(Q \):

\[Q = \frac{4}{3} \times \frac{3}{25} = \frac{4}{25}. \]

Particular Solution:

\[
Y_p = \frac{3}{25} \cos X + \frac{4}{25} \sin X.
\]

Summary of the General Solution

The complete solution to the differential equation combines the homogeneous and particular solutions:

$$
\boxed{
Y = (A + Bx) e^{-2x} + \frac{3}{25} \cos X + \frac{4}{25} \sin X,
}
$$

where A and B are arbitrary constants.

Additional Insights and Techniques

Handling Repeated Roots & Alternative Methods

- The characteristic root $r = -2$ is repeated, resulting in the homogeneous solution involving e^{-2x} and $x e^{-2x}$.
- When the right side of the differential equation involves functions similar to the homogeneous solution, the method of undetermined coefficients might need adjustment (e.g., multiplying the particular solution guess by x to avoid duplication). However, in this case, since $\cos X$ is not part of the homogeneous solution, the straightforward guess suffices.

Variation of Parameters

- For more complex nonhomogeneous terms, variation of parameters is an alternative method. It involves using the homogeneous solution to construct a particular solution by allowing the constants to be functions of X.
- For this specific equation, the method of undetermined coefficients is more straightforward and efficient.

Practical Applications & Significance

Understanding how to solve such differential equations is vital in multiple domains:

- Physics: Modeling forced oscillations, electrical circuits with sinusoidal inputs.
- Engineering: Analyzing mechanical systems under periodic forces.
- Mathematics: Building intuition about the interplay between homogeneous solutions and particular solutions.

Knowing how the method works deepens comprehension of the behavior of solutions, especially their asymptotic behavior, stability, and resonance phenomena.

Summary and Key Takeaways

- The homogeneous solution for $Y'' + 4Y' + 4Y = 0$ involves repeated roots,

leading to solutions of the form \((A + Bx) e^{rx} \).
- The particular solution for the forcing term \(\cos X \) can be guessed as a linear combination of sine and cosine functions.
- By substituting the particular guess into the differential equation, coefficients \(P \) and \(Q \) are determined through algebraic equations.
- The general solution combines both parts to offer a complete description of the system's behavior.

Final Remarks

Mastering the process of solving such second-order linear differential equations enhances problem-solving skills and prepares one

[Find The General Solution By Solving The Differential Equation Y 4y 4y Cos X Using The Method]

Find The General Solution By Solving The Differential Equation Y 4y 4y Cos X Using The Method

Related Articles

- In "Earthlings, Meet Venus," The Author Describes Venus' Unique Environment; In "All Summer In A Day,"
- In This Passage, It Can Be Inferred That Cassius Is Motivated To Use Brutuss Status To Convince Him To
- For Purposes Of Clarifying Ambiguities There Are Four Ways To Define A Word: By Definition Or Synonym,

Back to Home