Find The General Solution Of The Differential Equation Xt Dx/dt = X + 6t, Where T>0. A. X(t) = T3ln

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Understanding and solving differential equations is fundamental in mathematical analysis, physics, engineering, and many applied sciences. In this article, we will explore how to find the general solution of the differential equation:

$$ Xt \frac{dX}{dt} = X + 6t $$

where $T > 0$. We will also analyze the given expression $X(t) = T3 \ln$, interpret its significance, and demonstrate the step-by-step process to derive the general solution.

Introduction to Differential Equations

Differential equations involve functions and their derivatives, expressing how a quantity changes relative to another. They are categorized based on order, linearity, and whether they are ordinary or partial differential equations.

- Ordinary Differential Equations (ODEs): Involve functions of a single variable and their derivatives.
- Partial Differential Equations (PDEs): Involve functions of multiple variables and their partial derivatives.

The equation under consideration is an ordinary differential equation of the first order, involving $X(t)$ and its derivative dX/dt.

Analyzing the Given Differential Equation

The differential equation is:

$$ Xt \frac{dX}{dt} = X + 6t $$

Let's understand its structure:

- X is a function of t.
- The term $Xt \frac{dX}{dt}$ suggests a product involving X, t, and dX/dt.

Our goal is to find the general solution $X(t)$ that satisfies this equation for all $t > 0$.

Rearranging the Equation for Solving

To facilitate solving, rewrite the equation:

$$
Xt \frac{dX}{dt} = X + 6t
$$

Divide both sides by t:

$$
X \frac{dX}{dt} = \frac{X}{t} + 6
$$

This form suggests that substitution might help. Notice the structure involves X and dX/dt, which hints at a substitution involving X.

Substitution Method: Letting $Y = X^2$

Since the equation involves the product $X \frac{dX}{dt}$, observe that:

$$
X \frac{dX}{dt} = \frac{1}{2} \frac{d}{dt} (X^2)
$$

because:

$$
\frac{d}{dt} (X^2) = 2X \frac{dX}{dt}
$$

Thus, rewrite the differential equation as:

$$
\frac{1}{2} \frac{d}{dt} (X^2) = \frac{X}{t} + 6
$$

Multiply both sides by 2:

$$
\frac{d}{dt} (X^2) = 2 \frac{X}{t} + 12
$$

Recall that $Y = X^2$, so:

$$
\frac{dY}{dt} = 2 \frac{X}{t} + 12
$$

But since $Y = X^2$, then $X = \pm \sqrt{Y}$. To maintain generality, we proceed with X in terms of Y:

$$
\frac{dY}{dt} = 2 \frac{\sqrt{Y}}{t} + 12
$$

However, this involves the square root, which complicates solving directly. Alternatively, consider an integrating factor approach or substitution based on variables.

Transforming the Equation into a Bernoulli Equation

Let's revisit the original form to see if it can be transformed into a Bernoulli differential equation.

Original form:

$$
X t \frac{dX}{dt} = X + 6t
$$

Divide both sides by t:

$$
X \frac{dX}{dt} = \frac{X}{t} + 6
$$

Now, divide both sides by X (assuming $X \neq 0$):

$$
\frac{dX}{dt} = \frac{1}{t} + \frac{6}{X}
$$

This is a nonlinear differential equation involving X and dX/dt. Rearranged:

$$
\frac{dX}{dt} - \frac{6}{X} = \frac{1}{t}
$$

This is a Bernoulli equation in disguise but with a nonstandard form. Let's analyze further.

Solving the Differential Equation: Step-by-Step Approach

Given the previous manipulations, a more straightforward approach is to treat the original differential equation as a separable equation by appropriate substitution.

Step 1: Rewrite the original equation

$$
X t \frac{dX}{dt} = X + 6t
$$

Divide both sides by t:

$$
X \frac{dX}{dt} = \frac{X}{t} + 6
$$

Step 2: Recognize the structure

Recall that:

$$
X \frac{dX}{dt} = \frac{1}{2} \frac{d}{dt} (X^2)
$$

So, rewrite:

$$
\frac{1}{2} \frac{d}{dt} (X^2) = \frac{X}{t} + 6
$$

or

$$
\frac{d}{dt} (X^2) = 2 \frac{X}{t} + 12
$$

Step 3: Express in terms of $Y = X^2$

Let $Y = X^2$, then:

$$
\frac{dY}{dt} = 2 \frac{X}{t} + 12
$$

But since $X = \pm \sqrt{Y}$, the equation becomes:

$$
\frac{dY}{dt} = 2 \frac{\sqrt{Y}}{t} + 12
$$

This is a Bernoulli-type differential equation in $Y(t)$. To proceed, consider the substitution:

$$
Z = \sqrt{Y} \Rightarrow Y = Z^2
$$

then:

\[

$$
\frac{dY}{dt} = 2 Z \frac{dZ}{dt}
$$

Substitute back:

$$
2 Z \frac{dZ}{dt} = 2 \frac{Z}{t} + 12
$$

Divide both sides by 2:

$$
Z \frac{dZ}{dt} = \frac{Z}{t} + 6
$$

Now, divide both sides by Z (assuming $Z \neq 0$):

$$
\frac{dZ}{dt} = \frac{1}{t} + \frac{6}{Z}
$$

This is a Bernoulli differential equation in $Z(t)$:

$$
\frac{dZ}{dt} - \frac{1}{t} = \frac{6}{Z}
$$

Multiplying both sides by Z:

$$
Z \frac{dZ}{dt} - \frac{Z}{t} = 6
$$

Recall that $Z \frac{dZ}{dt} = \frac{1}{2} \frac{d}{dt} Z^2$:

$$
\frac{1}{2} \frac{d}{dt} Z^2 - \frac{Z}{t} = 6
$$

But perhaps it's more straightforward to stay with the previous form:

$$
\frac{dZ}{dt} - \frac{1}{t} = \frac{6}{Z}
$$

Solving the Bernoulli Equation in $Z(t)$

Let's write the differential equation explicitly:

$$
\frac{dZ}{dt} - \frac{1}{t} = \frac{6}{Z}
$$

Multiply both sides by Z:

$$
Z \frac{dZ}{dt} - \frac{Z^2}{t} = 6
$$

Note that:

$$
Z \frac{dZ}{dt} = \frac{1}{2} \frac{d}{dt} Z^2
$$

Thus:

$$
\frac{1}{2} \frac{d}{dt} Z^2 - \frac{Z^2}{t} = 6
$$

Let $W = Z^2$. Then:

$$
\frac{1}{2} \frac{dW}{dt} - \frac{W}{t} = 6
$$

Multiply through by 2:

$$
\frac{dW}{dt} - \frac{2W}{t} = 12
$$

This is a linear first-order differential equation in $W(t)$:

$$
\frac{dW}{dt} + P(t) W = Q(t)
$$

where:

- \(P

Frequently Asked Questions

What is the differential equation given in the problem?

The differential equation is Xt dX/dt = X + 6t, where T > 0.

How can we simplify the given differential equation for solving?

By dividing both sides by Tt, or rewriting in terms of standard variables, to convert it into a more manageable form such as a linear differential equation.

What method is suitable for solving the differential equation Xt dX/dt = X + 6t?

Using substitution or integrating factor methods suitable for linear first-order differential equations.

What is the general solution of the differential equation?

The general solution is $X(t) = C t^3 \ln(t) + t^3$, where C is an arbitrary constant.

How does the condition T > 0 influence the solution?

Since $T > 0$, it ensures that the logarithmic term $\ln(t)$ is defined for $t > 0$, making the solution valid in the domain $t > 0$.

Why is the form X(t) = T^3 ln(t) significant in the solution?

It indicates a particular solution involving a logarithmic term, highlighting the role of the natural logarithm in the solution structure due to the differential equation's form.

Additional Resources

Find The General Solution Of The Differential Equation $Xt \frac{dX}{dt} = X + 6t$, Where $T > 0$

Introduction

Differential equations serve as fundamental tools in modeling a vast array of phenomena across physics, engineering, biology, and economics. They describe how quantities change concerning one another, often capturing the essence of dynamic systems. Among these, first-order ordinary differential equations (ODEs) hold particular significance owing to their relative simplicity and broad applicability.

In this article, we delve into the problem of solving the differential equation:

$$
X t \frac{dX}{dt} = X + 6t
$$

where $X = X(t)$ is a function of t, and $T > 0$ is a parameter or constant involved in the problem context. Our goal is to find the general solution of this differential equation, thoroughly analyzing the steps, underlying concepts, and potential applications.

Understanding the Differential Equation

Before proceeding to solution strategies, it is crucial to interpret the structure and form of the differential equation:

$$
X t \frac{dX}{dt} = X + 6t
$$

Key observations:

- The equation involves the product $X t \frac{dX}{dt}$, implying a nonlinear relationship.
- The right side is linear in X and t.

Recasting the Equation

The first step in solving the differential equation involves rewriting it into a more manageable form.

Divide both sides of the equation by t:

$$
X \frac{dX}{dt} = \frac{X}{t} + 6
$$

This form suggests possible substitution strategies, especially considering the presence of the ratio $\frac{X}{t}$.

Substitution Strategy: Reducing to a Bernoulli Equation

Observe that the equation involves X and $\frac{X}{t}$. To simplify, consider the substitution:

$$
Y = \frac{X}{t}
$$

which implies:

$$
X = tY
$$

Differentiating $X = tY$ concerning t:

$$
\frac{dX}{dt} = Y + t \frac{dY}{dt}
$$

Substituting into the original differential equation:

$$
X \frac{dX}{dt} = tY \left(Y + t \frac{dY}{dt} \right)
$$

Recall that earlier, after dividing both sides by t, the equation was:

$$
X \frac{dX}{dt} = \frac{X}{t} + 6
$$

Replacing $X = tY$:

$$
tY \left(Y + t \frac{dY}{dt} \right) = Y t + 6
$$

Simplify the left:

$$
tY \cdot Y + tY \cdot t \frac{dY}{dt} = Y t + 6
$$

which becomes:

$$
t Y^2 + t^2 Y \frac{dY}{dt} = Y t + 6
$$

Subtract $Y t$ from both sides:

$$
t^2 Y \frac{dY}{dt} + t Y^2 - Y t = 6
$$

Notice that $t Y^2 - Y t = 0$, so the left simplifies to:

$$
t^2 Y \frac{dY}{dt} = 6
$$

Dividing both sides by $t^2 Y$:

$$
\frac{dY}{dt} = \frac{6}{t^2 Y}
$$

Deriving a Separable Differential Equation

The resulting equation:

$$
\frac{dY}{dt} = \frac{6}{t^2 Y}
$$

is a separable differential equation because variables Y and t are separable:

$$
Y \, dY = \frac{6}{t^2} dt
$$

Integrating Both Sides

Integrate both sides:

$$
\int Y \, dY = \int \frac{6}{t^2} dt
$$

The integrals are straightforward:

- Left side:

$$
\int Y \, dY = \frac{Y^2}{2} + C_1
$$

- Right side:

$$
\int \frac{6}{t^2} dt = 6 \int t^{-2} dt = 6 \left(- t^{-1} \right) + C_2 = - \frac{6}{t} + C_2
$$

Combining constants (say, C):

$$
\frac{Y^2}{2} = - \frac{6}{t} + C
$$

or equivalently:

$$
Y^2 = - \frac{12}{t} + 2C
$$

Let $K = 2C$, an arbitrary constant:

$$
Y^2 = - \frac{12}{t} + K
$$

Recall the substitution:

$$
Y = \frac{X}{t}
$$

Therefore:

$$
\left(\frac{X}{t}\right)^2 = - \frac{12}{t} + K
$$

Multiplying both sides by t^2:

$$
X^2 = - 12 t + K t^2
$$

This provides the general solution:

$$
\boxed{
X(t) = \pm \sqrt{K t^2 - 12 t}
}
$$

where K is an arbitrary constant determined by initial conditions.

Conditions for Validity and Domain

The expression under the square root must be non-negative:

$$
K t^2 - 12 t \geq 0
$$

which simplifies to:

$$
t (K t - 12) \geq 0
$$

Given that $t > 0$, the inequality depends on:

$$
K t - 12 \geq 0 \implies t \geq \frac{12}{K}
$$

assuming $K > 0$. Alternatively, for $K < 0$, the quadratic might be negative, invalidating the real solutions unless t satisfies specific conditions.

Summary of the Solution

The general solution of the differential equation:

$$
X t \frac{dX}{dt} = X + 6t
$$

is:

$$
\boxed{
X(t) = \pm \sqrt{K t^2 - 12 t}
}
$$

where K is an arbitrary real constant determined by initial conditions, with the domain constrained by the non-negativity of the expression under the square root.

Additional Insights and Implications

1. Behavior Near $t = 0$:

As $t \to 0^+$, the expression:

$$
X(t) \sim \pm \sqrt{-12 t}
$$

which is imaginary unless $K t^2 - 12 t \geq 0$, indicating potential singularities or domain restrictions at small t.

2. Particular Solutions:

Setting $K = 0$:

$$
X(t) = \pm \sqrt{-12 t}
$$

which is real only for $t \leq 0$, incompatible with the assumption $t > 0$. Thus, physically relevant solutions require $K > 0$.

3. Parameter Dependence:

The constant K encapsulates initial conditions, such as initial values of X at specific t.

Conclusion

The process of solving the differential equation $X t \frac{dX}{dt} = X + 6t$ illustrates a methodical approach: recognizing structure, applying substitution to reduce complexity, integrating, and analyzing the solution's domain.

The key steps involved:

- Recognizing the nonlinear structure.
- Using the substitution $Y = \frac{X}{t}$ to convert the original equation into a separable form.
- Integrating to find an explicit relation involving X and t.
- Interpreting the constant of integration and domain restrictions.

The resulting general solution:

$$
X(t) = \pm \sqrt{K t^2 - 12 t}
$$

provides a comprehensive understanding of the behavior of the solutions depending on initial conditions and parameter choices.

This detailed analysis not only offers a solution but also exemplifies critical techniques in differential equations, enriching the toolkit for tackling similar nonlinear problems encountered across scientific disciplines.

Find The General Solution Of The Differential Equation Xt Dx Dt X 6t Where T Gt 0 A X T T3ln

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