Find The Maximum And Minimum Values Of The Function F(x, Y) = 2x + 3y2 4x 5 On The Domain X2 + Y2 <

Find The Maximum And Minimum Values Of The Function $F(x, Y) = 2x + 3y^2 - 4x + 5$ On The Domain $X^2 + Y^2 <$

Understanding how to find the maximum and minimum values of a function within a specific domain is a fundamental concept in calculus and optimization. In this article, we will explore how to determine the maximum and minimum values of the function $F(x, y) = 2x + 3y^2 - 4x + 5$ on the domain defined by the inequality $X^2 + Y^2 < r^2$ (assuming the domain is the interior of a circle of radius r). This problem involves techniques such as critical point analysis and boundary examination, which are essential tools in multivariable calculus.

Introduction to the Problem

When working with functions of two variables, such as F(x, y), the goal is often to find the points within a specified domain where the function attains its highest and lowest values. These are known as the maximum and minimum, respectively. The domain in this case is the interior of a circle, defined by the inequality $X^2 + Y^2 < r^2$, where r is the radius.

The function in question is:

$$F(x, y) = 2x + 3y^2 - 4x + 5$$

which simplifies to:

$$F(x, y) = -2x + 3y^2 + 5$$

Now, our task is to find the maximum and minimum values of F(x, y) within the circle $X^2 + Y^2 < r^2$.

Step 1: Simplify the Function

Before proceeding, note that the function can be simplified:

- $F(x, y) = 2x + 3y^2 - 4x + 5$
- $F(x, y) = (-2x) + 3y^2 + 5$

This form makes it easier to analyze the behavior of the function.

Step 2: Find Critical Points Within the Domain

Critical points are points where the gradient of the function is zero. To find these points:

Partial Derivatives

Calculate the partial derivatives with respect to x and y:

- $\partial F/\partial x = -2$
- $\partial F/\partial y = 6y$

Set these derivatives equal to zero to find critical points:

- $\partial F/\partial x = -2 \neq 0$ — The derivative with respect to x is constant and never zero.
- $\partial F/\partial y = 6y = 0 \Rightarrow y = 0$

Since $\partial F/\partial x$ is never zero, there are no critical points inside the domain based on the gradient being zero. However, the function's behavior along the boundary and at certain points should still be examined.

Step 3: Analyze the Boundary of the Domain

The boundary of the domain is the circle:

$$x^2 + y^2 = r^2$$

Because the interior ($x^2 + y^2 < r^2$) does not contain critical points, the maximum and minimum values are attained either at boundary points or at points approaching the boundary.

Parametrize the Boundary

Using parametric equations:

- $x = r \cos \theta$

- $y = r \sin \theta$

where θ varies from 0 to 2π.

Express F(x, y) on the Boundary

Substituting the parametric form into F(x, y):

$$F(r \cos \theta, r \sin \theta) = -2(r \cos \theta) + 3(r \sin \theta)^2 + 5$$

Simplify:

$$F(\theta) = -2r \cos \theta + 3r^2 \sin^2 \theta + 5$$

Note that $\sin^2 \theta = (1 - \cos 2\theta)/2$, so:

$$F(\theta) = -2r \cos \theta + 3r^2 (1 - \cos 2\theta)/2 + 5$$

which simplifies to:

$$F(\theta) = -2r \cos \theta + (3r^2/2) - (3r^2/2) \cos 2\theta + 5$$

Step 4: Find the Extreme Values on the Boundary

To find the maximum and minimum of $F(\theta)$, analyze the function:

$$F(\theta) = -2r \cos \theta - (3r^2/2) \cos 2\theta + (3r^2/2) + 5$$

Now, express $\cos 2\theta$ in terms of $\cos \theta$:

Recall that $\cos 2\theta = 2 \cos^2 \theta - 1$. Thus:

$$F(\theta) = -2r \cos \theta - (3r^2/2)(2 \cos^2 \theta - 1) + (3r^2/2) + 5$$

Distribute:

$$F(\theta) = -2r \cos \theta - 3r^2 \cos^2 \theta + (3r^2/2) + (3r^2/2) + 5$$

Simplify:

$$F(\theta) = -2r \cos \theta - 3r^2 \cos^2 \theta + 3r^2 + 5$$

Now, define $x = \cos \theta$, where $x \in [-1, 1]$.

So, the problem reduces to analyzing the function:

$$G(x) = -2r x - 3r^2 x^2 + 3r^2 + 5, \text{ where } x \in [-1, 1]$$

Finding Extreme Values of G(x)

Differentiate G(x) with respect to x:

$dG/dx = -2r - 6r^2 x$

Set derivative to zero to find critical points:

$-2r - 6r^2 x = 0$

Solve for x:

$x = -2r / (6r^2) = -2r / (6r^2) = -1 / (3r)$

Check whether this critical point $x = -1 / (3r)$ lies within [-1, 1]:

- If $r > 1/3$, then $x = -1/(3r) \in (-1, 1)$, so the critical point is within domain.
- If $r \leq 1/3$, then $x = -1/(3r) \geq -1$, but may be outside the domain depending on r.

Evaluating G(x) at Critical Point and Endpoints

Calculate G(x) at $x = -1/(3r)$:

$G(-1/(3r)) = -2r (-1/(3r)) - 3r^2 (-1/(3r))^2 + 3r^2 + 5$

Simplify:

- First term: $-2r (-1/(3r)) = 2/3$

- Second term: $-3r^2 (1 / (9 r^2)) = -3r^2 (1 / (9 r^2)) = -1/3$

- Sum: $2/3 - 1/3 + 3r^2 + 5 = (1/3) + 3r^2 + 5$

Therefore:

$G(-1/(3r)) = (1/3) + 3r^2 + 5 = (16/3) + 3r^2$

Now, evaluate G(x) at the endpoints $x = -1$ and $x = 1$:

- For $x = -1$:

$G(-1) = -2r (-1) - 3r^2 1 + 3r^2 + 5 = 2r - 3r^2 + 3r^2 + 5 = 2r + 5$

- For $x = 1$:

$G(1) = -2r 1 - 3r^2 1 + 3r^2 + 5 = -2r - 3r^2 + 3r^2 + 5 = -2r + 5$

Summary of boundary evaluations:

- At $x = -1$: $G = 2r + 5$
- At $x = 1$: $G = -2r + 5$
- At critical point $x = -1/(3r)$: $G = (16/3) + 3r^2$

Step 5: Determine the Maximum and Minimum Values

Now, compare these values to find the maximum and minimum:

- The maximum among the boundary points:

- For $r > 0$, $2r + 5$ increases with r, so the maximum at $x = -1$ is $2r + 5$.

- The critical point gives $G = (16/3) + 3r^2$, which increases with r^2.

- The minimum:

- For $r > 0$, $-2r +$

Frequently Asked Questions

What is the function we are analyzing for maximum and minimum values?

The function is F(x, y) = 2x + 3y^2 - 4x + 5.

What is the domain specified for the function F(x, y)?

The domain is the set of points where x^2 + y^2 < 1, i.e., the interior of the unit circle.

How do we find the critical points of the function within the

domain?

By calculating the partial derivatives of F with respect to x and y, setting them equal to zero, and solving for x and y.

What are the partial derivatives of F(x, y)?

The partial derivatives are: $\partial F/\partial x = 2 - 4$ and $\partial F/\partial y = 6y$.

What do the critical points tell us about the maximum and minimum values?

Critical points indicate potential locations of local maxima or minima; these points need to be checked within the domain.

Are there any critical points inside the domain x^2 + y^2 < 1?

Yes, solving the derivatives shows that the critical point occurs at $x = -1$, $y = 0$, which lies outside the interior of the unit circle.

How do we evaluate the maximum and minimum values on the boundary of the domain?

By parametrizing the boundary x^2 + y^2 = 1 and evaluating F at those points to find extrema.

What is the parametrization of the boundary circle?

The boundary can be parametrized as $x = \cos\theta$, $y = \sin\theta$, where θ ranges from 0 to 2π.

How do we find the maximum and minimum values of F on the boundary?

Substitute $x = \cos\theta$ and $y = \sin\theta$ into F to get a function of θ, then analyze this function to find its maximum and minimum.

What is the final step to determine the maximum and minimum of F within the domain?

Compare the critical points' values (if any inside the domain) with the boundary evaluation results to identify the global maximum and minimum within x^2 + y^2 < 1.

Additional Resources

Optimization Analysis of the Function $F(x, y) = 2x + 3y^2 - 4x - 5$ on the Domain $x^2 + y^2 < 1$

Introduction

In the realm of multivariable calculus, understanding the maximum and minimum values of a function within a specified domain is fundamental—be it for academic pursuits, engineering design, or economic modeling. Today, we delve into an intriguing investigation: finding the maximum and minimum values of the function

$$F(x, y) = 2x + 3y^2 - 4x - 5$$

over the unit disk domain characterized by

$$x^2 + y^2 < 1.$$

This exploration isn't just a dry mathematical exercise; it’s a comprehensive journey through the principles of optimization, boundary analysis, and critical point identification, all crucial skills for any mathematician or engineer.

Simplifying the Function

Before diving into optimization techniques, it's advantageous to simplify the function:

$$F(x, y) = 2x + 3y^2 - 4x - 5.$$

Combine like terms:

$$F(x, y) = (2x - 4x) + 3y^2 - 5 = -2x + 3y^2 - 5.$$

This simplified form provides clarity:

$$F(x, y) = -2x + 3y^2 - 5.$$

The function exhibits linearity in x (with a slope of -2) and quadratic dependence in y (with a positive coefficient, indicating a parabola opening upward in y^2). The constant term -5 shifts the entire surface downward.

The Domain: The Unit Disk

The domain $x^2 + y^2 < 1$ describes the interior of a circle with radius 1 centered at the origin. This domain is open, meaning it does not include the boundary $x^2 + y^2 = 1$, but for maximum and minimum value purposes, the extremal points will occur either inside or on the boundary.

In applied problems, often the boundary points are critical, especially when the interior does not contain any critical points that maximize or minimize the function.

Approach to Finding Extrema

The process involves two main steps:

1. Identify critical points within the interior of the domain by setting the gradient of F to zero.
2. Examine the boundary $x^2 + y^2 = 1$ to find potential maxima or minima, since on a bounded domain, the extrema can occur either at critical points or on the boundary.

Critical Points in the Interior

Step 1: Compute the gradient of $F(x, y)$

$$
\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right).
$$

Calculating each component:

$$
\frac{\partial F}{\partial x} = -2,
$$
$$
\frac{\partial F}{\partial y} = 6y.
$$

Step 2: Find points where the gradient is zero

Set both partial derivatives to zero:

$$
-2 = 0 \quad \Rightarrow \quad \text{No solution},
$$
$$
6y = 0 \quad \Rightarrow \quad y = 0.
$$

Since $-2 \neq 0$, there are no critical points inside the domain where both partial derivatives vanish simultaneously. The gradient indicates the function has a constant slope in x, suggesting the function is strictly increasing or decreasing in the x-direction, and only critical points may exist along the boundary or at infinity (which is outside our domain).

Conclusion: No interior critical points exist; the function's extremal values occur on the boundary.

Boundary Analysis: The Circle $x^2 + y^2 = 1$

Since the interior lacks critical points, we focus on the boundary $x^2 + y^2 = 1$. To analyze F on this circle, we can parameterize the boundary.

Step 3: Parameterization of the boundary

Let:

$$
x = \cos \theta, \quad y = \sin \theta,
$$
where $\theta \in [0, 2\pi]$.

Substitute into $F(x, y)$:

$$
F(\theta) = -2\cos \theta + 3 \sin^2 \theta - 5.
$$

Recall $\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$, so:

$$
F(\theta) = -2\cos \theta + 3 \times \frac{1 - \cos 2\theta}{2} - 5,
$$
$$
F(\theta) = -2\cos \theta + \frac{3}{2} - \frac{3}{2} \cos 2\theta - 5,
$$
$$
F(\theta) = -2\cos \theta - \frac{3}{2} \cos 2\theta - \frac{7}{2}.
$$

Now, $F(\theta)$ is expressed in terms of $\cos \theta$ and $\cos 2\theta$, which are straightforward to analyze.

Simplifying the Boundary Function

Express $\cos 2\theta$ using the double angle formula:

$$
\cos 2\theta = 2 \cos^2 \theta - 1.
$$

Substitute into $F(\theta)$:

$$
F(\theta) = -2 \cos \theta - \frac{3}{2} (2 \cos^2 \theta - 1) - \frac{7}{2},
$$
\[

$$
F(\theta) = -2 \cos \theta - 3 \cos^2 \theta + \frac{3}{2} - \frac{7}{2},
$$
$$
F(\theta) = -2 \cos \theta - 3 \cos^2 \theta - 2.
$$

Define $t = \cos \theta$, with the domain $-1 \leq t \leq 1$. The problem reduces to finding extrema of:

$$
F(t) = -2t - 3 t^2 - 2,
$$
for $t \in [-1, 1]$.

Optimization over $[-1, 1]$

Step 4: Find critical points of $F(t)$

Compute derivative with respect to t:

$$
F'(t) = -2 - 6t.
$$

Set $F'(t) = 0$:

$$
-2 - 6t = 0,
$$
$$
6t = -2,
$$
$$
t = -\frac{1}{3}.
$$

Since $-\frac{1}{3}$ lies within $[-1, 1]$, it is a valid critical point.

Step 5: Evaluate $F(t)$ at critical points and endpoints

- At $t = -\frac{1}{3}$:

$$
F\left(-\frac{1}{3}\right) = -2 \times \left(-\frac{1}{3}\right) - 3 \times \left(-\frac{1}{3}\right)^2 - 2,
$$
$$
= \frac{2}{3} - 3 \times \frac{1}{9} - 2,
$$

$$
= \frac{2}{3} - \frac{1}{3} - 2,
$$

$$
= \left(\frac{2}{3} - \frac{1}{3}\right) - 2 = \frac{1}{3} - 2 = -\frac{5}{3}.
$$

- At $t = 1$:

$$
F(1) = -2(1) - 3(1)^2 - 2 = -2 - 3 - 2 = -7.
$$

- At $t = -1$:

$$
F(-1) = -2(-1) - 3(1) - 2 = 2 - 3 - 2 = -3.
$$

Summary of Boundary Extrema

t	$F(t)$	Interpretation
-1	-3	Boundary point θ with $\cos \theta = -1$ $\Rightarrow$ $\theta = \pi$, $x = -1$, $y=0$.
$-\frac{1}{3}$	$-\frac{5}{3}$	Critical point inside $[-1,1]$. Corresponds to some θ with $\cos \theta = -1/3$.
1	-7	Boundary point with $\cos \theta = 1$ $\Rightarrow$ $\theta = 0$, $x=1$, $y=0$.

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