Find The Order Of Growth Of The Following Entire Functions: (a) P(z) Where P Is A Polynomial. (b) Ebz"

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Understanding the growth behavior of entire functions is a fundamental aspect of complex analysis. Entire functions are those complex functions that are holomorphic over the entire complex plane. Their growth rates as |z| approaches infinity provide deep insights into their nature and classification. In this article, we explore the order of growth for two particular types of entire functions: polynomials and the exponential of a linear function, specifically the functions P(z), where P is a polynomial, and Ebz". We will analyze these functions comprehensively, providing definitions, properties, and detailed derivations to understand their growth behaviors.

Introduction to Entire Functions and Growth Rates

Before delving into specific functions, it's essential to establish foundational concepts related to entire functions and their growth rates.

What Are Entire Functions?

Entire functions are complex functions that are holomorphic at every point in the complex plane $\mathbb{C}$. Examples include polynomials, exponential functions, sine, cosine, and many special functions. These functions are distinguished by their global analyticity, meaning they have power series expansions convergent everywhere.

Order and Type of Entire Functions

The order of an entire function quantifies the overall growth rate of the function as $|z| \to \infty$. The type further refines this measure within a given order.

Definition (Order):
For an entire function $f(z)$, the order ρ is defined as:

$$
\rho = \limsup_{r \to \infty} \frac{\ln \ln M(r)}{\ln r}
$$

where

$$
M(r) = \max_{|z|=r} |f(z)|
$$

is the maximum modulus of $f(z)$ on the circle of radius r.

- If ρ is finite, the function grows no faster than some exponential of r^{ρ}.
- If $\rho = 0$, the function is of order zero, indicating very slow growth.
- If $\rho = \infty$, the function grows faster than any finite order.

Type σ: For functions of finite order ρ, the type is given by:

$$
\sigma = \limsup_{r \to \infty} \frac{\ln M(r)}{r^{\rho}}
$$

which measures the growth rate within the order.

Growth of Polynomial Functions

Let's begin with the simplest class of entire functions: polynomials.

Polynomial Functions $P(z)$

Suppose $P(z)$ is a polynomial of degree n:

$$
P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0
$$

where $a_n \neq 0$.

Growth Behavior of Polynomials

Polynomials are among the most well-understood entire functions. Their growth rate as $|z| \to \infty$ is polynomial, which is significantly slower than exponential functions.

Maximum Modulus for Polynomials:

$$
M(r) = \max_{|z|=r} |P(z)| \approx |a_n| r^n \quad \text{for large } r
$$

since the highest degree term dominates as $|z|$ becomes large.

Implications for the Order:

Using the definition:

$$
\rho = \limsup_{r \to \infty} \frac{\ln \ln M(r)}{\ln r}
$$

we substitute $M(r) \sim |a_n| r^n$:

$$
\ln M(r) \sim \ln |a_n| + n \ln r
$$

then,

$$
\ln \ln M(r) \sim \ln (n \ln r)
$$

which for large r behaves as:

$$
\ln \ln M(r) \sim \ln n + \ln \ln r
$$

Now, compute the order:

$$
\rho = \limsup_{r \to \infty} \frac{\ln n + \ln \ln r}{\ln r} = \lim_{r \to \infty} \frac{\ln \ln r}{\ln r}

$$

Since $\ln \ln r$ grows much more slowly than $\ln r$, the ratio tends to zero:

$$
\rho = 0
$$

Conclusion:

$$
\boxed{
\text{The order of polynomial functions } P(z) \text{ is } 0
}
$$

This result confirms that polynomials grow at a rate slower than any exponential function, classifying them as entire functions of order zero.

Growth of the Function $E_b z"$

Now, let's analyze the second function, which appears to be an exponential type function, possibly indicating the exponential of a linear function or some related form. Assuming the notation $E_b z"$ refers to $E_b(z) = e^{b z}$, where b is a constant, or similarly, an exponential function involving z.

Exponential Functions of the Form $E_b(z) = e^{b z}$

The exponential function $e^{b z}$ is a canonical example of an entire function with exponential growth.

Maximum Modulus for $E_b(z)$:

$$
M(r) = \max_{|z|=r} |e^{b z}|
$$

Given $b \in \mathbb{C}$, write $b = |b| e^{i \theta}$. For $|z| = r$, the maximum modulus occurs when $z = r e^{i \phi}$ aligned with b:

$$
|e^{b z}| = e^{\Re(b z)} = e^{r \Re(b e^{i \phi})}
$$

which attains its maximum when $\phi = -\theta$, thus:

$$
|e^{b z}| \leq e^{|b| r}
$$

Growth Rate and Order:

Using the maximum modulus:

$$
M(r) \sim e^{|b| r}
$$

then,

$$
\ln M(r) \sim |b| r
$$

and

$$
\ln \ln M(r) \sim \ln (|b| r) = \ln |b| + \ln r
$$

Calculating the order:

$$
\rho = \limsup_{r \to \infty} \frac{\ln \ln M(r)}{\ln r} = \lim_{r \to \infty} \frac{\ln |b| + \ln r}{\ln r} = 1
$$

since $\lim_{r \to \infty} \frac{\ln r}{\ln r} = 1$.

Therefore:

$$
\boxed{
\text{The exponential function } e^{b z} \text{ has order } 1

}
\]

and, as expected, exponential functions are of order one.

Summary of Results

Function Type	Growth Rate	Order	Explanation
Polynomial $P(z)$	Polynomial growth	0	Because $M(r) \sim r^n$, very slow compared to exponentials
Exponential $e^{b z}$	Exponential growth	1	Because $M(r) \sim e^{|b| r}$, characteristic of order 1

Implications and Applications of Growth Orders

Understanding the order of entire functions is crucial in various areas of complex analysis and its applications.

- Classifying Entire Functions: The order helps categorize functions into classes like finite order, order zero, or infinite order.
- Value Distribution Theory: Growth order influences the behavior of zeros, Picard's theorems, and value distribution.
- Differential Equations: Solutions to linear differential equations often have growth rates linked to their order.
- Approximation Theory: Polynomial approximations are more feasible for functions of finite order, especially order zero.

Conclusion

In summary, the growth characteristics of entire functions are essential in understanding their behavior at infinity and their classification within complex analysis.

- For polynomials \(P(z)\), the growth is polynomial, and the order is zero.
- For exponential functions \(e^{b z}\), the growth is exponential, and the order is one.

Recognizing these fundamental differences aids in advanced studies involving entire functions, such as Nevanlinna theory, asymptotic analysis, and the theory of entire solutions to differential equations.

Further Reading and Resources

- Complex Analysis by Elias M. Stein and Rami Shakarchi
- Entire and Meromorphic Functions by R. P. Boas
- Introduction to the Theory of Entire Functions by R. P. Boas
- Online resources

Frequently Asked Questions

What is the order of growth of a polynomial function P(z)?

The order of growth of a polynomial P(z) is finite and equals its degree, meaning it grows polynomially as |z| approaches infinity.

How do entire functions like Ebz compare in growth to polynomials?

Functions like Ebz, which are exponential functions, have a much faster, exponential order of growth compared to polynomials, which grow polynomially.

What is the formal definition of the order of an entire function?

The order of an entire function f(z) is defined as the infimum of positive numbers ρ such that $|f(z)| \leq \exp(|z|^{\rho})$ for sufficiently large |z|.

What is the order of growth of a polynomial P(z)?

The order of growth of a polynomial P(z) is 0, because polynomials grow at a rate less than any exponential function; their growth is finite and polynomial.

How does the exponential function Ebz influence the order of growth?

The exponential function Ebz grows exponentially with respect to |z|, giving it an order of growth equal to 1, which is faster than any polynomial.

Are entire functions with finite order necessarily polynomials?

No, entire functions with finite order include polynomials and certain transcendental functions like the exponential, but not all transcendental functions have finite order.

What is the order of Ebz as |z| approaches infinity?

The order of Ebz is 1, because its growth rate is exponential in |z|.

Why is understanding the order of growth important in complex analysis?

It helps classify entire functions based on their growth behavior at infinity, which is crucial for understanding their properties, value distribution, and in the application of theorems like the Hadamard factorization.

Additional Resources

Find The Order Of Growth Of The Following Entire Functions: (a) P(z) Where P Is A Polynomial. (b) Ebz

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Introduction

In complex analysis, understanding the growth behavior of entire functions is fundamental for classifying functions based on their asymptotic properties as |z| approaches infinity. Entire functions are complex functions that are holomorphic across the entire complex plane. Their growth rates are typically described using the concepts of order and type, which quantify how rapidly a function's magnitude increases.

This piece aims to analyze and determine the order of growth of two specific classes of entire functions:

- (a) Polynomial functions, denoted as $P(z)$, where P is a polynomial.
- (b) The exponential function e^{bz}, where $b \in \mathbb{C}$.

We will explore the definitions of order and type, their computational methods, and then apply these to the given functions to determine their growth behaviors comprehensively.

Fundamental Concepts: Order and Type of Entire Functions

What is the Order of an Entire Function?

The order ρ of an entire function $f(z)$ measures the function’s maximum possible rate of growth as $|z| \to \infty$. Formally, it is defined as:

$$
\rho = \limsup_{r \to \infty} \frac{\ln \ln M(r)}{\ln r}
$$

where:

- $M(r) = \max_{|z|=r} |f(z)|$ is the maximum modulus of f on the circle of radius r.

Intuitively:

- If f grows no faster than $\exp(r^\alpha)$ for some α, then the order ρ is at most α.
- Conversely, if f grows faster than any such $\exp(r^\alpha)$, then the order is higher.

The order essentially classifies functions into classes based on their asymptotic growth:

Order ρ	Growth Behavior	Typical Examples
0	Bounded or sub-exponential growth	Constant functions, polynomials
$0 < \rho < 1$	Sub-exponential but faster than polynomial	Certain entire functions of order less than 1
1	Exponential growth	e^{z}, $e^{a z}$
$\rho > 1$	Faster-than-exponential growth	$\exp(\exp z)$ etc.

What is the Type of an Entire Function?

For functions with a fixed order ρ, the type σ refines the classification by quantifying the precise exponential growth rate within that order.

Given that f has order ρ, the type is defined as:

\[

$$
\sigma = \limsup_{r \to \infty} \frac{\ln M(r)}{r^\rho}
$$

- Type 0 indicates slow growth within the order class.
- For example, for order 1 functions, the type measures how "close" the growth is to pure exponential growth.

Analytical Methods for Determining Order

To find the order of a specific entire function $f(z)$:

1. Compute $M(r)$: The maximum modulus on the circle $|z| = r$.

2. Evaluate the limit superior:

$$
\rho = \limsup_{r \to \infty} \frac{\ln \ln M(r)}{\ln r}
$$

3. Apply known growth behaviors:

- Polynomial functions $P(z)$ grow like $|z|^{n}$, so their order is 0.
- Exponential functions of the form $e^{b z}$ grow like $e^{|b| r}$, so their order is 1.
- More complex functions may require asymptotic analysis or known entire function classifications.

Part (a): Polynomial Functions $P(z)$

Definition and Basic Properties

Let $P(z)$ be a polynomial of degree n:

$$
P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0
$$

where $a_n \neq 0$.

Growth Behavior of Polynomials

- The maximum modulus $M(r)$ for $P(z)$ behaves like:

$$
M(r) \sim |a_n| r^n \quad \text{as} \quad r \to \infty
$$

- Since $M(r) \sim C r^n$, for some constant $C > 0$, taking logarithms yields:

$$
\ln M(r) \sim \ln C + n \ln r
$$

- Further,

$$
\ln \ln M(r) \sim \ln (n \ln r + \text{constant}) \sim \ln n + \ln \ln r
$$

- Therefore,

$$
\frac{\ln \ln M(r)}{\ln r} \sim \frac{\ln n + \ln \ln r}{\ln r} \to 0 \quad \text{as} \quad r \to \infty
$$

Conclusion for Polynomials

$$
\boxed{
\text{Order of } P(z) \text{ is } 0
}
$$

Rationale: Polynomial functions grow slower than exponential functions. Their maximum modulus increases polynomially, and the double logarithm of $M(r)$ grows much more slowly than $\ln r$, leading to an order of zero.

Part (b): The Exponential Function $e^{b z}$

Definition and Basic Properties

- For $b \in \mathbb{C}$, the exponential function $e^{b z}$ is an entire function.

- The magnitude on the circle $|z|=r$ is:

$$
|e^{b z}| = e^{\Re(b z)} \leq e^{|b| r}
$$

since $|\Re(b z)| \leq |b| r$.

Growth Behavior of $e^{b z}$

- The maximum modulus $M(r)$ satisfies:

$$
M(r) \sim e^{|b| r}
$$

- Taking logarithms:

$$
\ln M(r) \sim |b| r
$$

- The double logarithm:

$$
\ln \ln M(r) \sim \ln |b| r = \ln |b| + \ln r
$$

- Compute the order:

$$
\rho = \limsup_{r \to \infty} \frac{\ln \ln M(r)}{\ln r} = \lim_{r \to \infty} \frac{\ln r + \text{constant}}{\ln r} = 1
$$

Conclusion for $e^{b z}$:

$$
\boxed{
\text{Order of } e^{b z} \text{ is } 1
}
$$

Rationale: The exponential function grows exactly like $e^{|b| r}$, which corresponds to order 1.

Summary of Results

Function	Growth Order	Description	Remarks
Polynomial $P(z)$	0	Sub-exponential, polynomial growth	All polynomials are order zero.
$e^{b z}$	1	Exponential growth	Classic exponential growth, order one.

Additional Insights and Examples

Examples Demonstrating the Concepts

1. Constant function $f(z) = c$:

- $M(r) = |c|$
- Growth order = 0 (bounded).

2. Quadratic polynomial $P(z) = z^2$:

- $M(r) \sim r^2$
- Growth order = 0.

3. Exponential with real coefficient e^{z}:

- $M(r) \sim e^{r}$
- Growth order = 1.

4. Super-exponential $\exp(\exp z)$:

- $M(r) \sim e^{e^{r}}$
- This grows faster than any exponential; its order is infinite.

Broader Implications and Applications

Understanding the order of entire functions is crucial in several areas:

- Value distribution theory: The order influences the frequency and distribution of zeros.
- Approximation theory: Polynomial and exponential functions serve as building blocks for approximations, with growth rates dictating convergence.
- Complex differential equations: Solutions often involve entire functions, and their growth behavior affects stability and asymptotic analysis.
- Transcendental number theory: The order helps classify transcendental functions and their properties.

Summary

In conclusion, the growth order of entire functions serves as a foundational classification in complex analysis:

- Polynomials are order zero, reflecting their

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