Find The Point On The Plane 3x + 4y + Z = 8 That Is Nearest To (2, 0, 1). What Are The Values Of X, Y,

Find The Point On The Plane 3x + 4y + Z = 8 That Is Nearest To (2, 0, 1). What Are The Values Of X, Y, is a classic problem in analytic geometry that involves finding the closest point on a given plane to a specified point in three-dimensional space. This problem is not only fundamental in understanding the geometric relationships between points and planes but also has practical applications in fields such as engineering, computer graphics, and optimization. In this comprehensive guide, we will explore the methods to determine the point on the plane 3x + 4y + z = 8 that minimizes the distance to the point (2, 0, 1). We will delve into the mathematical concepts involved, step-by-step solutions, and relevant tips to enhance your understanding.

Understanding the Problem: Finding the Closest Point on a Plane

Before diving into the solution, it’s essential to understand the core aspects of the problem:

- Given Plane Equation: 3x + 4y + z = 8
- Given Point: (2, 0, 1)
- Objective: Find the point (x, y, z) on the plane that is closest to (2, 0, 1)

This problem can be summarized as an optimization problem where we aim to minimize the distance between the point (2, 0, 1) and any point (x, y, z) lying on the plane.

Mathematical Foundations for the Solution

To find the point on the plane closest to a given point, we can utilize concepts from vector calculus and geometry:

Distance Between Two Points

The Euclidean distance between points (x, y, z) and (x_0, y_0, z_0) is given by:

\[
D = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}
\]

In our case, the point $(x_0, y_0, z_0) = (2, 0, 1)$.

Minimizing the Distance

Since the square root function is monotonically increasing, minimizing the distance D is equivalent to minimizing the squared distance:

$$
D^2 = (x - 2)^2 + (y - 0)^2 + (z - 1)^2
$$

This approach simplifies calculations, as it avoids dealing with the square root.

Constrained Optimization

The point must satisfy the plane equation:

$$
3x + 4y + z = 8
$$

This is a constraint that must be incorporated into the optimization problem, which can be tackled using the method of Lagrange multipliers.

Step-by-Step Solution: Finding the Nearest Point

Let's now go through the process of solving the problem systematically:

Step 1: Set Up the Objective Function

Define the squared distance function:

$$
f(x, y, z) = (x - 2)^2 + (y)^2 + (z - 1)^2
$$

Our goal is to minimize $f(x, y, z)$ subject to the constraint:

$$
g(x, y, z) = 3x + 4y + z - 8 = 0
$$

Step 2: Apply Lagrange Multipliers

Construct the Lagrangian:

$$
\mathcal{L}(x, y, z, \lambda) = f(x, y, z) - \lambda g(x, y, z)
$$

which expands to:

$$
\mathcal{L} = (x - 2)^2 + y^2 + (z - 1)^2 - \lambda (3x + 4y + z - 8)
$$

Step 3: Find Partial Derivatives and Set Them to Zero

Calculate the partial derivatives:

$$
\frac{\partial \mathcal{L}}{\partial x} = 2(x - 2) - 3\lambda = 0
$$
$$
\frac{\partial \mathcal{L}}{\partial y} = 2y - 4\lambda = 0
$$
$$
\frac{\partial \mathcal{L}}{\partial z} = 2(z - 1) - \lambda = 0
$$
$$
\frac{\partial \mathcal{L}}{\partial \lambda} = -(3x + 4y + z - 8) = 0
$$

This gives us a system of equations:

1. $2(x - 2) = 3\lambda$ → $x - 2 = \frac{3}{2}\lambda$
2. $2y = 4\lambda$ → $y = 2\lambda$
3. $2(z - 1) = \lambda$ → $z - 1 = \frac{\lambda}{2}$
4. $3x + 4y + z = 8$

Solving the System of Equations

Let's solve for x, y, z in terms of λ:

- From equation 1:

$$
x = 2 + \frac{3}{2}\lambda

\]

- From equation 2:

\[
y = 2\lambda
\]

- From equation 3:

\[
z = 1 + \frac{\lambda}{2}
\]

Substitute these into the constraint equation:

\[
3x + 4y + z = 8
\]

which becomes:

\[
3 \left(2 + \frac{3}{2}\lambda \right) + 4(2\lambda) + \left(1 + \frac{\lambda}{2} \right) = 8
\]

Simplify step-by-step:

\[
3 \times 2 + 3 \times \frac{3}{2}\lambda + 8\lambda + 1 + \frac{\lambda}{2} = 8
\]
\[
6 + \frac{9}{2}\lambda + 8\lambda + 1 + \frac{\lambda}{2} = 8
\]

Combine like terms:

\[
(6 + 1) + \left(\frac{9}{2}\lambda + 8\lambda + \frac{\lambda}{2} \right) = 8
\]
\[
7 + \left(\frac{9}{2}\lambda + 8\lambda + \frac{\lambda}{2} \right) = 8
\]

Express all \(\lambda\) terms with denominator 2:

\[
\frac{9}{2}\lambda + \frac{16}{2}\lambda + \frac{1}{2}\lambda = \left(9 + 16 + 1 \right)/2 \lambda = 26/2 \lambda = 13 \lambda
\]

Now, the equation simplifies to:

$$
7 + 13 \lambda = 8
$$

Solve for λ:

$$
13 \lambda = 8 - 7 = 1
$$

$$
\lambda = \frac{1}{13}
$$

Step 4: Find the Coordinates of the Closest Point

Substitute $\lambda = \frac{1}{13}$ back into the expressions for x, y, z:

- $x = 2 + \frac{3}{2} \times \frac{1}{13} = 2 + \frac{3}{26} = 2 + \frac{3}{26} = \frac{52}{26} + \frac{3}{26} = \frac{55}{26}$

- $y = 2 \times \frac{1}{13} = \frac{2}{13}$

- $z = 1 + \frac{1}{13} \times \frac{1}{2} = 1 + \frac{1}{26} = \frac{26}{26} + \frac{1}{26} = \frac{27}{26}$

Thus, the point on the plane closest to (2, 0, 1) is:

$$
\boxed{
\left(\frac{55}{26}, \frac{2}{13}, \frac{27}{26} \right)
}
$$

Interpreting the Results

The obtained point $\left(\frac{55}{26}, \frac{2}{13}, \frac{27}{26} \right)$ is the unique point on the plane 3x + 4y + z = 8 that minimizes the Euclidean distance to the point (2, 0, 1). Let’s analyze the significance of this point:

- Coordinates in decimal form:

$$
x \approx 2.115, \quad y \approx 0.154, \quad z \approx 1.038
$$

- Distance calculation:

To verify this is the closest point, the Euclidean distance between (2

Frequently Asked Questions

How do I find the point on the plane 3x + 4y + z = 8 that is closest to the point (2, 0, 1)?

To find the closest point, you can use the method of projecting the point onto the plane along the line perpendicular to it, which involves finding the foot of the perpendicular from (2, 0, 1) to the plane.

What is the formula to determine the closest point on a plane to a given point in 3D space?

The closest point can be found using the formula: $P_{closest} = P - ((P - P0) \cdot N / N \cdot N) N$, where P0 is the given point, N is the normal vector of the plane, and · denotes the dot product.

What is the normal vector of the plane 3x + 4y + z = 8?

The normal vector N of the plane is (3, 4, 1).

How do I compute the scalar projection needed to find the closest point on the plane?

Calculate the dot product of the vector from the point to a point on the plane with the normal vector, then divide by the magnitude squared of the normal vector to find the scalar projection.

What are the steps to find the coordinates (x, y, z) of the nearest point on the plane to (2, 0, 1)?

First, find the normal vector, then compute the vector from the point to the plane, project this onto the normal, and subtract this projection from the original point to get the nearest point's coordinates.

What are the specific values of x and y for the point on

the plane closest to (2, 0, 1)?

The closest point on the plane to (2, 0, 1) has coordinates approximately x ≈ 1.68, y ≈ -0.84, z ≈ 3.24, determined by applying the projection method.

Additional Resources

Find The Point On The Plane 3x + 4y + z = 8 That Is Nearest To (2, 0, 1): An In-Depth Analytical Exploration

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Introduction: The Significance of Distance Minimization in Geometry

In the realm of analytical geometry, understanding how to determine the closest point on a plane to a specified external point is a fundamental problem with applications spanning fields such as physics, engineering, computer graphics, and optimization. This problem not only tests our grasp of geometric principles but also underscores the power of algebraic methods in solving spatial challenges.

Specifically, the question posed—"Find the point on the plane 3x + 4y + z = 8 that is nearest to the point (2, 0, 1)"—serves as an excellent example of how to approach such problems systematically. It demands a blend of geometric intuition and algebraic rigor to arrive at an accurate solution.

This article aims to dissect and analyze this problem comprehensively, beginning with foundational concepts, progressing through detailed solution methodologies, and finally interpreting the implications of the results.

Understanding the Geometric Context

The Plane Equation and Its Geometric Representation

The given plane is described by the linear equation:

$$ 3x + 4y + z = 8 $$

This is a standard form of a plane in three-dimensional space, where the coefficients of x, y, and z (namely, 3, 4, and 1 respectively) define the plane's orientation, and the constant term (8) determines its position relative to the origin.

The vector normal to this plane, often called the "normal vector," is derived directly from the coefficients of x, y, and z:

$$ \mathbf{n} = (3, 4, 1) $$

This normal vector is perpendicular to every vector lying within the plane, making it a crucial element in the analysis of perpendicular distances.

The External Point (2, 0, 1)

The point from which we measure the shortest distance to the plane is given as:

$$ \mathbf{P}_0 = (2, 0, 1) $$

Understanding the position of this point relative to the plane helps us visualize the problem: the goal is to find the point $\mathbf{Q} = (x, y, z)$ on the plane such that the Euclidean distance between $\mathbf{P}_0$ and $\mathbf{Q}$ is minimized.

The Mathematical Foundation: Distance from a Point to a Plane

Euclidean Distance in Three Dimensions

The Euclidean distance between two points $\mathbf{A} = (x_1, y_1, z_1)$ and $\mathbf{B} = (x_2, y_2, z_2)$ is given by:

$$ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} $$

In our problem, $\mathbf{A}$ is the point $\mathbf{P}_0 = (2, 0, 1)$, and $\mathbf{Q} = (x, y, z)$ is a generic point on the plane.

The task reduces to finding the point $\mathbf{Q}$ on the plane such that the distance $d(\mathbf{P}_0, \mathbf{Q})$ is minimized.

Optimizing Distance via Projection

A key insight in solving such problems is that the shortest distance from a point to a plane occurs along the line perpendicular (normal) to the plane, passing through the point.

This geometrical fact implies that the minimal distance point $\mathbf{Q}$ lies along the line passing through $\mathbf{P}_0$ in the direction of the normal vector $\mathbf{n}$.

The problem then reduces to:

- Finding the point $\mathbf{Q}$ on the plane that is the orthogonal projection of $\mathbf{P}_0$ onto the plane.
- Ensuring that $\mathbf{Q}$ lies on the plane $3x + 4y + z = 8$.

Methodology: Step-by-Step Solution

Step 1: Parametric Equation of the Line Passing Through $\mathbf{P}_0$ Along $\mathbf{n}$

Since the shortest distance from $\mathbf{P}_0$ to the plane occurs along the normal, we consider the line passing through $\mathbf{P}_0$ in the direction of $\mathbf{n} = (3, 4, 1)$.

The parametric form of this line is:

$$ \mathbf{L}(t) = \mathbf{P}_0 + t \mathbf{n} = (2 + 3t, 0 + 4t, 1 + t) $$

where t is a real parameter.

Step 2: Find the Intersection of the Line with the Plane

To find the point on the line that lies on the plane, substitute the parametric coordinates into the plane equation:

$$ 3x + 4y + z = 8 $$

Plugging in:

$$ 3(2 + 3t) + 4(4t) + (1 + t) = 8 $$

Simplify:

$$ 6 + 9t + 16t + 1 + t = 8 $$

Combine like terms:

$$ (6 + 1) + (9t + 16t + t) = 8 $$

$$ 7 + 26t = 8 $$

Solve for t:

$$ 26t = 8 - 7 $$
$$ 26t = 1 $$
$$ t = \frac{1}{26} $$

This value of t indicates the specific point along the line that intersects the plane perpendicularly from $\mathbf{P}_0$.

Step 3: Calculate the Coordinates of the Closest Point $\mathbf{Q}$

Using $t = \frac{1}{26}$, compute the coordinates:

$$ x = 2 + 3 \times \frac{1}{26} = 2 + \frac{3}{26} = \frac{52}{26} + \frac{3}{26} = \frac{55}{26} $$

$$ y = 0 + 4 \times \frac{1}{26} = \frac{4}{26} = \frac{2}{13} $$

$$ z = 1 + 1 \times \frac{1}{26} = 1 + \frac{1}{26} = \frac{26}{26} + \frac{1}{26} = \frac{27}{26} $$

Therefore, the point $\mathbf{Q}$ on the plane closest to $\mathbf{P}_0$ is:

$$ \boxed{
\left(\frac{55}{26}, \frac{2}{13}, \frac{27}{26} \right)
} $$

Interpreting the Results: Values of x, y, and z

The explicit values obtained—fractions with precise denominators—highlight the exact nature of the solution. For practical purposes, approximate decimal equivalents are:

- $ x \approx 2.115 $
- $ y \approx 0.154 $
- $ z \approx 1.038 $

These coordinates define the point on the plane that minimizes the Euclidean distance to $\mathbf{P}_0$.

Verifying the Solution: Geometric and Algebraic Checks

Distance Calculation

To confirm the minimality, we can compute the distance between $\mathbf{P}_0 = (2, 0, 1)$ and $\mathbf{Q} = \left(\frac{55}{26}, \frac{2}{13}, \frac{27}{26}\right)$:

$$ d = \sqrt{\left(2 - \frac{55}{26}\right)^2 + \left(0 - \frac{2}{13}\right)^2 + \left(1 - \frac{27}{26}\right)^2} $$

Express each difference with common denominators:

- $2 = \frac{52}{26}$, so:

$$ 2 - \frac{55}{26} = \frac{52}{26} - \frac{55}{26} = -\frac{3}{26} $$

- $0 = \frac{0}{13} = \frac{0}{13}$, but to match denominators, it's better to convert all to denominator 26:

$$ \frac{0}{13} = \frac{0}{13} \Rightarrow \text{but better to convert } \frac{2}{13} = \frac{4}{26} $$

Thus,

$$ 0 - \frac{2}{13} = 0 - \frac{4}{26} = -\frac{4}{26} $$

Similarly

Find The Point On The Plane 3x 4y Z 8 That Is Nearest To 2 0 1 What Are The Values Of X Y

Find The Point On The Plane 3x 4y Z 8 That Is Nearest To 2 0 1 What Are The Values Of X Y

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