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Understanding the behavior of second-order systems is fundamental in control engineering, signal processing, and many related fields. When designing or analyzing such systems, the placement of poles in the s-plane (complex plane) directly influences the system's transient response, stability, and performance characteristics. One key parameter often specified is the overshoot, which indicates how much the system output exceeds its final steady-state value during transient response.

In this article, we will explore how to determine the location of poles for a second-order system given a specific overshoot value, in this case, 10%. We will delve into the relationships between overshoot, damping ratio, natural frequency, and pole locations, providing a comprehensive guide to solving this problem effectively.

Understanding Second-Order Systems and Their Parameters

What is a Second-Order System?

A second-order system is a dynamic system characterized by a differential equation of second order. Such systems are common in mechanical, electrical, and aerospace engineering applications. The standard form of a second-order transfer function is:

$$
H(s) = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2}
$$

where:

- ω_n = natural (undamped) frequency (rad/sec)
- ζ = damping ratio (unitless)

The poles of this transfer function are the roots of the denominator:

$$
s^2 + 2\zeta \omega_n s + \omega_n^2 = 0
$$

which are given by:

\[

$$
s_{1,2} = -\zeta \omega_n \pm j \omega_n \sqrt{1 - \zeta^2}
$$

The real part $-\zeta \omega_n$ determines the rate of exponential decay, while the imaginary part $\omega_n \sqrt{1 - \zeta^2}$ relates to oscillation frequency.

Relationship Between Overshoot and Damping Ratio

What is Overshoot?

Percent overshoot (OS) measures how much the transient response exceeds the steady-state value during its initial oscillation. It’s expressed as a percentage:

$$
OS = \frac{C_{peak} - C_{ss}}{C_{ss}} \times 100\%
$$

where:

- C_{peak} = maximum response value
- C_{ss} = steady-state response value

For a second-order underdamped system, the overshoot is directly related to the damping ratio:

$$
OS = e^{-\frac{\zeta \pi}{\sqrt{1 - \zeta^2}}} \times 100\%
$$

or in decimal form:

$$
OS_{decimal} = e^{-\frac{\zeta \pi}{\sqrt{1 - \zeta^2}}}
$$

Note: The overshoot is valid for $\zeta < 1$ (underdamped systems).

Calculating Damping Ratio for 10% Overshoot

Given:

$$
OS = 10\%
$$

Converting to decimal:

$$
OS_{decimal} = 0.10
$$

Using the overshoot relation:

$$
0.10 = e^{-\frac{\zeta \pi}{\sqrt{1 - \zeta^2}}}
$$

Taking natural logarithm on both sides:

$$
\ln(0.10) = -\frac{\zeta \pi}{\sqrt{1 - \zeta^2}}
$$

$$
-2.302585 = -\frac{\zeta \pi}{\sqrt{1 - \zeta^2}}
$$

$$
2.302585 = \frac{\zeta \pi}{\sqrt{1 - \zeta^2}}
$$

Rearranged to solve for ζ:

$$
\frac{\zeta}{\sqrt{1 - \zeta^2}} = \frac{2.302585}{\pi} \approx
\frac{2.302585}{3.141593} \approx 0.733
$$

Let:

$$
x = \zeta
$$

then:

$$
\frac{x}{\sqrt{1 - x^2}} = 0.733
$$

Squaring both sides:

$$
\frac{x^2}{1 - x^2} = 0.733^2 \approx 0.537
$$

Cross-multiplied:

$$
x^2 = 0.537 (1 - x^2)
$$

$$
x^2 = 0.537 - 0.537 x^2

$$
x^2 + 0.537 x^2 = 0.537
$$

$$
(1 + 0.537) x^2 = 0.537
$$

$$
1.537 x^2 = 0.537
$$

$$
x^2 = \frac{0.537}{1.537} \approx 0.349
$$

$$
x = \zeta = \sqrt{0.349} \approx 0.59
$$

Therefore, the damping ratio for a 10% overshoot is approximately $\zeta \approx 0.59$.

Determining the Pole Locations in the Complex Plane

Poles of a Second-Order System

The poles are located at:

$$
s_{1,2} = -\zeta \omega_n \pm j \omega_n \sqrt{1 - \zeta^2}
$$

Given $\zeta \approx 0.59$, the only remaining variable is the natural frequency ω_n. Typically, in control system design, the natural frequency is chosen based on other transient response specifications such as rise time, settling time, or bandwidth.

Assumption: If no specific ω_n is provided, the poles are expressed parametrically in terms of ω_n.

Expressing the Pole Locations

Substitute $\zeta \approx 0.59$:

$$
s_{1,2} = -0.59 \omega_n \pm j \omega_n \sqrt{1 - 0.59^2}
$$

Calculate $\sqrt{1 - 0.59^2}$:

$$
0.59^2 = 0.3481
$$

$$
\sqrt{1 - 0.3481} = \sqrt{0.6519} \approx 0.8074
$$

Therefore, the poles are:

$$
s_{1,2} = -0.59 \omega_n \pm j 0.8074 \omega_n
$$

This indicates that the poles are located at:

- Real part: $-0.59 \omega_n$
- Imaginary part: $\pm 0.8074 \omega_n$

Note: The magnitude of the pole locations depends on ω_n. For specific response characteristics, ω_n can be selected accordingly.

Implications for System Design and Response

Choosing ω_n Based on Response Speed

The natural frequency ω_n influences the speed of the transient response:

- Higher ω_n results in faster response.
- Lower ω_n yields a slower response but potentially less overshoot if ζ remains the same.

For example, if the desired settling time T_s (to within 2%) is specified, it relates to ζ and ω_n:

$$
T_s \approx \frac{4}{\zeta \omega_n}
$$

Suppose a settling time of 1 second is desired:

$$
\omega_n = \frac{4}{\zeta T_s} = \frac{4}{0.59 \times 1} \approx 6.78 \text{ rad/sec}
$$

then the pole locations are approximately:

$$
s_{1,2} = -0.59 \times 6.78 \pm j 0.8074 \times 6.78 \approx -4.00 \pm j 5.47
$$

which are located at:

$$
s_{1,2} \approx -4.00 \pm j 5.47
$$

Summary: How to Find the Pole Locations for 10% Overshoot

Step-by-step process:

1. Determine damping ratio (ζ) from overshoot:

- Use the relation $OS = e^{-\frac{\zeta \pi}{\sqrt{1 - \zeta^2}}}$.
- For 10% overshoot, $\zeta \approx 0.59$.

2. Choose or determine the natural frequency (ω_n):

- Based on transient response requirements such as settling time or rise time.
- For example, for a settling time \(T

Frequently Asked Questions

How do you determine the damping ratio for a second-order system with an overshoot of 10%?

The damping ratio ζ can be found using the overshoot formula: $OS = e^{-(\zeta\pi)/(\sqrt{(1-\zeta^2)})}$. For a 10% overshoot, solve for ζ: $0.10 = e^{-(\zeta\pi)/(\sqrt{(1-\zeta^2)})}$, which yields $\zeta \approx 0.591$.

What is the relationship between damping ratio and pole locations in a second-order system?

The damping ratio ζ determines how far the poles are from the real axis in the s-plane. Poles are located at $s = -\zeta\omega_n \pm j\omega_n\sqrt{(1-\zeta^2)}$, where ω_n is the natural frequency. A higher ζ indicates poles closer to the negative real axis, resulting in less oscillation.

How do you find the natural frequency ω_n given an overshoot of 10% and a known damping ratio?

Once ζ is known (approximately 0.591 for 10% overshoot), ω_n can be calculated if the system's settling time or other parameters are available. If not, ω_n remains a variable; the pole locations are primarily determined by ζ and ω_n, with the real part being $-\zeta\omega_n$ and the imaginary part $\omega_n\sqrt{(1-\zeta^2)}$.

What are the approximate pole locations for a second-order system with 10% overshoot?

Assuming a damping ratio $\zeta \approx 0.591$ and a natural frequency ω_n (chosen based on desired speed), the poles are at $s = -\zeta\omega_n \pm j\omega_n\sqrt{(1-\zeta^2)}$. For example, if $\omega_n = 10$ rad/sec, then the poles are approximately at $s = -5.91 \pm j7.65$.

How does increasing the damping ratio affect the pole locations and overshoot?

Increasing the damping ratio moves the poles closer to the negative real axis, reducing the imaginary component and decreasing overshoot. Poles become more real and less oscillatory, leading to a more overdamped response.

Can the location of the poles be directly calculated from the overshoot alone for a second-order system?

Yes, by first determining the damping ratio from the overshoot, then using the natural frequency, the pole locations can be expressed as $s = -\zeta\omega_n \pm j\omega_n\sqrt{(1-\zeta^2)}$. Without ω_n, only the relative location (based on ζ) can be specified.

Additional Resources

Second-Order System Poles and Overshoot Analysis: An In-Depth Exploration

Understanding the dynamic behavior of control systems is fundamental to designing stable and efficient engineering solutions. Among various system classifications, second-order systems hold particular significance due to their prevalence in practical applications such as robotics, aerospace, and electrical engineering. A critical aspect of analyzing these systems involves determining the location of their poles—complex values that dictate stability, transient response, and steady-state behavior. When the overshoot is specified, for instance at 10%, it provides valuable insights into the damping characteristics and consequently the pole placement. This article delves into the analytical process of locating poles in a second-order system given an overshoot of 10%, offering a comprehensive understanding suitable for students, researchers, and practitioners alike.

Understanding Second-Order Systems

Definition and Standard Form

A second-order system is characterized by a differential equation of the form:

$$\frac{d^2 y(t)}{dt^2} + 2 \zeta \omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 u(t)$$

where:
- $y(t)$ is the system output,
- $u(t)$ is the input (often a step input),
- ω_n is the natural frequency,
- ζ is the damping ratio.

The transfer function of a standard second-order system with unity feedback is expressed as:

$$G(s) = \frac{\omega_n^2}{s^2 + 2 \zeta \omega_n s + \omega_n^2}$$

The denominator polynomial's roots (the system's poles) are:

$$s_{1,2} = -\zeta \omega_n \pm j \omega_n \sqrt{1 - \zeta^2}$$

Poles are complex conjugates whose real part determines stability and damping, while the imaginary part influences oscillation frequency.

Role of Overshoot in System Response

Defining Overshoot

Overshoot (OS) measures how much the response exceeds its final steady-state value during transient oscillation, expressed as a percentage:

$$\text{Overshoot} = \frac{y_{max} - y_{ss}}{y_{ss}} \times 100\%$$

where:
- y_{max} is the maximum transient response,
- y_{ss} is the steady-state value.

In underdamped second-order systems, overshoot is directly related to the damping ratio, making it a key parameter in pole placement.

Link Between Overshoot and Damping Ratio

The overshoot percentage relates to the damping ratio ζ through the formula:

$$OS(\%) = 100 \times e^{\left(- \frac{\zeta \pi}{\sqrt{1 - \zeta^2}} \right)}$$

Rearranged to solve for ζ:

$$\zeta = - \frac{\ln(OS/100)}{\sqrt{\pi^2 + [\ln(OS/100)]^2}}$$

Given an overshoot of 10%, the damping ratio can be precisely calculated.

Calculating the Damping Ratio for 10% Overshoot

Step-by-step Calculation

Given:
- $OS = 10\%$

Calculate:
- $\ln(OS/100) = \ln(0.10) \approx -2.3026$

Plug into the damping ratio formula:

$$
\zeta = - \frac{-2.3026}{\sqrt{\pi^2 + (-2.3026)^2}}
$$

Calculate denominator:

$$
\sqrt{\pi^2 + (2.3026)^2} = \sqrt{(3.1416)^2 + (2.3026)^2} = \sqrt{9.8696 + 5.3019} = \sqrt{15.1715} \approx 3.895
$$

Thus:

$$
\zeta \approx \frac{2.3026}{3.895} \approx 0.591
$$

Result: For a 10% overshoot, the damping ratio ζ is approximately 0.591.

Determining the Pole Locations

Relationship Between Damping Ratio, Natural Frequency, and Pole Location

The poles for a second-order underdamped system are:

$$
s_{1,2} = -\zeta \omega_n \pm j \omega_n \sqrt{1 - \zeta^2}
$$

To locate the poles definitively, the natural frequency ω_n must be specified or chosen based on desired response speed.

Note: Without a specified ω_n, the poles can only be expressed in terms of ω_n. The key is that the real part of the poles is:

$$
\sigma = - \zeta \omega_n
$$

and the imaginary part:

$$
\omega_d = \omega_n \sqrt{1 - \zeta^2}
$$

Choosing the Natural Frequency ω_n

While the damping ratio determines the shape of the transient response (overshoot and damping), the natural frequency influences the speed of response:

- A higher ω_n results in a faster response.
- The poles move further to the left in the s-plane (more negative real parts), increasing stability and reducing settling time.

For illustrative purposes, suppose the system's desired settling time or bandwidth is known, or a typical value is selected based on application requirements.

Practical Example: Specifying ω_n and Locating the Poles

Suppose the designer desires a speed of response such that the system's settling time T_s (to within 2%) is approximately 1 second.

The approximate relation:

$$
T_s \approx \frac{4}{\zeta \omega_n}
$$

Rearranged to find ω_n:

$$
\omega_n = \frac{4}{\zeta T_s} = \frac{4}{0.591 \times 1} \approx 6.77\, \text{rad/sec}
$$

Now, the poles are:

$$
s_{1,2} = -0.591 \times 6.77 \pm j \times 6.77 \times \sqrt{1 - 0.591^2}
$$

Calculate real part:

$$
\sigma = -0.591 \times 6.77 \approx -4.00
$$

Calculate imaginary part:

$$
\omega_d = 6.77 \times \sqrt{1 - 0.591^2} = 6.77 \times \sqrt{1 - 0.349} = 6.77 \times \sqrt{0.651} \approx 6.77 \times 0.807 \approx 5.46
$$

Pole locations:

$$
s_{1,2} \approx -4.00 \pm j 5.46
$$

These complex conjugate poles define the transient response with approximately 10% overshoot and a settling time around 1 second.

Implications for System Design and Stability

Pole placement directly influences system behavior:

- Location: Poles farther left in the s-plane (more negative real parts) produce faster transient response but may increase control effort or actuator saturation.
- Damping ratio: Determines overshoot and oscillation. For 10% overshoot, a damping ratio of approximately 0.59 strikes a balance between responsiveness and minimal overshoot.
- Natural frequency: Adjusted based on desired response speed, with higher ω_n leading to quicker responses but potentially higher control energy.

Furthermore, the stability criterion for second-order systems is that all poles must have negative real parts, which is satisfied here with $\sigma = -4$.

Analytical Summary and System Design Approach

To analyze or design a second-order system with a specified overshoot:

1. Identify the desired overshoot and compute the damping ratio ζ using the formula:

$$
\zeta = - \frac{\ln(OS/100)}{\sqrt{\pi^2 + (\ln(OS/100))^2}}
$$

2. Determine the system's natural frequency ω_n based on transient response specifications such as settling time or bandwidth.

3. Calculate the pole locations:

$$
s_{1,2} = - \zeta \omega_n \pm j \omega_n \sqrt{1 - \zeta^2}
$$

4. Verify system stability by ensuring both poles have negative real parts.

5. Adjust ω_n as per system requirements to achieve the desired speed and damping characteristics.

Conclusion: Bridging Theory and Practical Design

The process of locating poles in a second-order system based on overshoot specifications exemplifies the harmony between theoretical control principles and

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