Given F(x) = 3^2x Cos 3x^2, X0 = 0.5 Use Newton Formula To Compute The Approximations X1, X2 And X3 Of

Given $F(x) = 3^{2x} \cos 3x^2$, $X_0 = 0.5$ Use Newton Formula To Compute The Approximations X_1, X_2 And X_3 Of

---

Introduction to Newton-Raphson Method and Its Application

In numerical analysis, solving nonlinear equations is a fundamental task with numerous applications across engineering, physics, and mathematics. When an explicit solution for an equation like $F(x) = 0$ is difficult or impossible to obtain analytically, iterative methods such as the Newton-Raphson method provide an efficient approach to approximate roots with high precision.

The Newton-Raphson method leverages the derivative of the function to iteratively improve guesses of the root. Its quadratic convergence near the root makes it particularly powerful for functions that are smooth and differentiable. In our context, we are given a specific function:

$$
F(x) = 3^{2x} \cos 3x^2
$$

with an initial approximation $X_0 = 0.5$. Our goal is to use the Newton-Raphson formula to compute successive approximations $X_1, X_2,$ and X_3 to the root of $F(x) = 0$.

Understanding the Function $F(x) = 3^{2x} \cos 3x^2$

Before applying the Newton-Raphson method, it's essential to analyze the function:

- Components of $F(x)$:
- 3^{2x}: An exponential function with base 3 raised to an exponent

$2x$.
- $\cos 3x^2$: A cosine function with quadratic argument $3x^2$.

- Behavior of $F(x)$:
- 3^{2x} increases exponentially as $x \to \infty$ and approaches zero as $x \to -\infty$.
- $\cos 3x^2$ oscillates rapidly for large x, with zeros at points where $3x^2 = \frac{\pi}{2} + k\pi$.

Given the initial guess $X_0 = 0.5$, which is close to zero, we expect the root to be near this point, but the oscillatory nature of the cosine term suggests careful derivative calculations are necessary.

Applying the Newton-Raphson Method

The Newton-Raphson iteration formula is:

$$
X_{n+1} = X_n - \frac{F(X_n)}{F'(X_n)}
$$

where:
- X_n is the current approximation,
- $F(X_n)$ is the function evaluated at X_n,
- $F'(X_n)$ is the derivative of F at X_n.

Our task involves computing:
1. X_1 using $X_0 = 0.5$,
2. X_2 using X_1,
3. X_3 using X_2.

Step-by-Step Calculation of Derivatives

To implement the Newton-Raphson method, we need the derivative $F'(x)$.

Given:
$$
F(x) = 3^{2x} \cos 3x^2
$$

Let's differentiate:

1. Rewrite 3^{2x} as an exponential:

$$
3^{2x} = e^{2x \ln 3}
$$

so that:

$$
\frac{d}{dx} 3^{2x} = 3^{2x} \cdot 2 \ln 3
$$

2. Using the product rule:

$$
F'(x) = \frac{d}{dx} \left(3^{2x} \cdot \cos 3x^2 \right) = \left(\frac{d}{dx} 3^{2x} \right) \cdot \cos 3x^2 + 3^{2x} \cdot \frac{d}{dx} \left(\cos 3x^2 \right)
$$

3. Calculate each term:

- First term:

$$
\left(3^{2x} \cdot 2 \ln 3 \right) \cdot \cos 3x^2
$$

- Second term:

$$
3^{2x} \cdot \left(- \sin 3x^2 \cdot 6x \right)
$$

(since $\frac{d}{dx} \cos 3x^2 = - \sin 3x^2 \cdot 6x$)

4. Final derivative:

$$
F'(x) = 3^{2x} \left[2 \ln 3 \cdot \cos 3x^2 - 6x \sin 3x^2 \right]
$$

Numerical Computation of Approximations

Now, we proceed to calculate the successive approximations:

Step 1: Compute X_1

Given:

$$
X_0 = 0.5
$$

Calculate $F(0.5)$:

$$
F(0.5) = 3^{2 \times 0.5} \cos 3 \times (0.5)^2 = 3^{1} \cos 3 \times 0.25 = 3 \times \cos 0.75
$$

Using $\cos 0.75 \approx 0.7317$:

$$
F(0.5) \approx 3 \times 0.7317 = 2.1951
$$

Calculate $F'(0.5)$:

$$
F'(0.5) = 3^{1} \left[2 \ln 3 \times \cos 0.75 - 6 \times 0.5 \times \sin 0.75 \right]
$$

$\ln 3 \approx 1.0986$

$\cos 0.75 \approx 0.7317$

$\sin 0.75 \approx 0.6816$

Compute:

$$
F'(0.5) = 3 \left[2 \times 1.0986 \times 0.7317 - 3 \times 0.6816 \right]
$$

$$
= 3 \left[2 \times 1.0986 \times 0.7317 - 2.0448 \right]
$$

Calculate:

$$
2 \times 1.0986 \times 0.7317 \approx 2 \times 0.804 \approx 1.608
$$

So:

$$
F'(0.5) \approx 3 \times (1.608 - 2.0448) = 3 \times (-0.4368) = -1.3104
$$

Now, apply Newton-Raphson:

$$
X_1 = 0.5 - \frac{2.1951}{-1.3104} \approx 0.5 + 1.674 \approx 2.174
$$

Step 2: Compute \(X_2\)

Using \(X_1 \approx 2.174\):

Calculate \(F(2.174)\):
- \(2 \times 2.174 = 4.348\),
- \(3^{4.348} = e^{4.348 \times 1.0986} \approx e^{4.775} \approx 119.5\),
- \(\cos 3 \times (2.174)^2 = \cos 3 \times 4.727 \approx \cos 14.181\).

\(\cos 14.181\):

- Since \(\cos\) is periodic with period \(2\pi \approx 6.283\),
- \(14.181 - 2 \times 6.283 = 14.181 - 12.566 = 1.615\),
- \(\cos 1.615 \approx -0.039\).

Thus:

\[
F(2.174) \approx 119.5 \times -0.039 \approx -4.66
\]

Calculate \(F'(2.174)\):

\[
F'(2.174) = 119.5 \left[2 \times 1.0986 \times \cos 14.181 - 6 \times 2.174 \times \sin 14.181 \right]
\]

Calculate:

- \(\cos 14.181 \approx -0.039\),
- \(\sin 14.181\):

Similarly, \(\sin 14.181\):

- \(14.181 - 2 \times 6.283 = 1.615\),
- \(\sin 1.615 \approx 0.999\).

Compute:

\[
F'(2.174) \approx 119.5 \left[2 \times 1.0986 \times (-0.039) - 6 \times 2.174 \times 0

Frequently Asked Questions

What is the given function F(x) in the problem?

The function is $F(x) = 3^{2x} \cos(3x^2)$.

What is the initial approximation X_0 provided in the problem?

The initial approximation is $X_0 = 0.5$.

How is Newton's method applied to find successive approximations X_1, X_2, and X_3?

Using Newton's formula: $X_{n+1} = X_n - F(X_n) / F'(X_n)$, where F' is the derivative of F.

What is the first step in computing X_1 using Newton's method?

Calculate F(0.5) and F'(0.5), then compute $X_1 = 0.5 - F(0.5)/F'(0.5)$.

How do you find the derivative F'(x) of the given function?

Apply the product and chain rules: F'(x) = derivative of $[3^{2x} \cos(3x^2)]$.

What is the significance of iterative approximations X_1, X_2, and X_3?

They progressively improve the estimate of the root of $F(x) = 0$ using Newton's method.

Are there any specific challenges in computing derivatives for exponential-trigonometric functions like F(x)?

Yes, combining exponential and trigonometric functions requires careful application of product and chain rules to accurately compute derivatives.

How can I verify the accuracy of the approximations X_1, X_2, and X_3?

By calculating $F(X_n)$ at each approximation; if the value is close to zero, the approximation is accurate, or monitor the difference $|X_{n+1} - X_n|$ for convergence.

Additional Resources

Given $F(x) = 3^{2x} \cos 3x^2$, $X_0 = 0.5$ Use Newton Formula To Compute The Approximations X_1, X_2 And X_3 Of

---

When tackling nonlinear equations, especially those involving exponential and trigonometric functions like $F(x) = 3^{2x} \cos 3x^2$, the Newton-Raphson method offers a powerful iterative approach to approximate roots. Starting with an initial guess $X_0 = 0.5$, we can employ the Newton formula to progressively refine our estimates, arriving at better approximations $X_1, X_2,$ and X_3. This blog post provides a comprehensive, step-by-step guide on how to implement this process, including detailed calculations, derivative evaluations, and insights into the convergence behavior of the method.

Understanding the Problem Setup

Before diving into computations, it's essential to understand the core components of the problem:

- Function: $F(x) = 3^{2x} \cos 3x^2$
- Initial guess: $X_0 = 0.5$
- Objective: Use Newton's method to compute successive approximations $X_1, X_2,$ and X_3

The Newton-Raphson Method in Brief

The Newton-Raphson iterative formula is given by:

$$
X_{n+1} = X_n - \frac{F(X_n)}{F'(X_n)}
$$

where:

- $F(X_n)$ is the function value at the current approximation
- $F'(X_n)$ is the derivative of the function evaluated at X_n

This method relies on the idea of tangent lines: at each step, the tangent to the curve at X_n is used to approximate the root.

Step 1: Deriving $F'(x)$

Given the function:

$$
F(x) = 3^{2x} \cos 3x^2
$$

1.1 Rewrite the function for clarity

Note that:

$$
3^{2x} = e^{2x \ln 3}
$$

Thus:

$$
F(x) = e^{2x \ln 3} \cdot \cos 3x^2
$$

1.2 Compute $F'(x)$ using the product rule

$$
F'(x) = \frac{d}{dx} \left(e^{2x \ln 3} \cdot \cos 3x^2 \right)
$$

Applying the product rule:

$$
F'(x) = \left(\frac{d}{dx} e^{2x \ln 3} \right) \cdot \cos 3x^2 + e^{2x \ln 3} \cdot \frac{d}{dx} \left(\cos 3x^2 \right)
$$

1.3 Derivative of $e^{2x \ln 3}$

$$
\frac{d}{dx} e^{2x \ln 3} = 2 \ln 3 \cdot e^{2x \ln 3}
$$

1.4 Derivative of $\cos 3x^2$

$$
\frac{d}{dx} \cos 3x^2 = - \sin 3x^2 \cdot \frac{d}{dx} (3x^2) = - \sin 3x^2 \cdot 6x
$$

1.5 Final expression for $F'(x)$:

$$
F'(x) = 2 \ln 3 \cdot e^{2x \ln 3} \cdot \cos 3x^2 - 6x \cdot e^{2x \ln 3} \cdot \sin 3x^2

\]

or, factoring out \(e^{2x \ln 3}\):

\[
F'(x) = e^{2x \ln 3} \left(2 \ln 3 \cdot \cos 3x^2 - 6x \sin 3x^2 \right)
\]

Step 2: Evaluate \(F(x)\) and \(F'(x)\) at \(X_0 = 0.5\)

Let's compute these step-by-step.

2.1 Compute \(F(0.5)\):

\[
F(0.5) = 3^{2 \times 0.5} \cos 3 \times (0.5)^2
\]

\[
= 3^{1} \cos 3 \times 0.25
\]

\[
= 3 \times \cos 0.75
\]

Using \(\cos 0.75 \approx 0.7317\):

\[
F(0.5) \approx 3 \times 0.7317 = 2.1951
\]

2.2 Compute \(F'(0.5)\):

\[
F'(0.5) = e^{2 \times 0.5 \ln 3} \left(2 \ln 3 \cdot \cos 3 \times (0.5)^2 - 6 \times 0.5 \sin 3 \times (0.5)^2 \right)
\]

Calculate each component:

- \(2 \times 0.5 \ln 3 = 1 \times \ln 3 \approx 1.0986\)
- \(e^{1.0986} \approx 3\) (since \(e^{\ln 3} = 3\))
- \(\cos 0.75 \approx 0.7317\)
- \(\sin 0.75 \approx 0.6816\)

Now,

\[

$$
F'(0.5) \approx 3 \times \left(2 \times 1.0986 \times 0.7317 - 6 \times 0.5 \times 0.6816 \right)
$$

Calculate inside the brackets:

- $2 \times 1.0986 \times 0.7317 \approx 2 \times 0.804 \approx 1.608$
- $6 \times 0.5 \times 0.6816 = 3 \times 0.6816 \approx 2.0448$

Putting it all together:

$$
F'(0.5) \approx 3 \times (1.608 - 2.0448) = 3 \times (-0.4368) \approx -1.3104
$$

Step 3: Compute X_1

Using Newton's formula:

$$
X_1 = X_0 - \frac{F(X_0)}{F'(X_0)} = 0.5 - \frac{2.1951}{-1.3104}
$$

$$
X_1 \approx 0.5 + 1.675 \approx 2.175
$$

First approximation:

$$
X_1 \approx 2.175
$$

Step 4: Compute X_2

Now, evaluate $F(2.175)$ and $F'(2.175)$:

4.1 $F(2.175)$:

$$
F(2.175) = 3^{2 \times 2.175} \cos 3 \times (2.175)^2
$$

Calculate exponents:

- $2 \times 2.175 = 4.35$

- $3^{4.35} = e^{4.35 \ln 3}$

Since $\ln 3 \approx 1.0986$:

$$
4.35 \times 1.0986 \approx 4.778
$$

$$
3^{4.35} \approx e^{4.778} \approx \exp(4.778) \approx 119.37
$$

Calculate the argument of cosine:

$$
3 \times (2.175)^2 = 3 \times 4.7326 \approx 14.198
$$

$\cos 14.198$:

- $14.198 \text{ radians} \approx 14.198 - 2\pi \times 2 \approx 14.198 - 12.566 \approx 1.632$

$\cos 1.632 \approx -0.065$

Thus,

$$
F(2.175) \approx 119.37 \times (-0.065) \approx -7.76
$$

4.2 $F'(2.175)$:

Calculate components:

- $2 \times 2.175 \times \ln 3 = 4.35 \times 1.0986 \approx 4.778$
- $e^{4.778} \approx 119.37$
- $\cos 14.198 \approx -0.065$
- $\sin 14.198$:

Since \(14.198 - 2\pi \

Given F X 3 2x Cos 3x 2 X0 0 5 Use Newton Formula To Compute The Approximations X1 X2 And X3 Of

Given F X 3 2x Cos 3x 2 X0 0 5 Use Newton Formula To Compute The Approximations X1 X2 And X3 Of

Related Articles

- Is Based On Personal Knowledge Or Observation Of The Person Testifying. No Inference Or Presumption Is
- Joaquin Mows The Lawn At His Grandmothers Home During The Summer Months. Joaquin Measured The Distance
- Identify The Problem With Pronoun Reference As Ambiguous, Remote, Or Broad . When Jill Picked Up Nancy

Back to Home