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Understanding the properties of ellipses is fundamental in many fields such as mathematics, engineering, physics, and computer graphics. One common problem involves calculating the length of an arc of an ellipse, particularly within specific quadrants. In this article, we will explore how to determine the length of the arc of the first quadrant of the ellipse given by the equation:

$$ 9x^2 + 16y^2 = 144 $$

Our goal is to provide a comprehensive, step-by-step approach to calculating this arc length, covering the necessary mathematical background, methods, and interpretations to make the process accessible and insightful.

Understanding the Ellipse Equation

Before diving into the arc length calculation, it's essential to understand the form and properties of the given ellipse.

The Standard Form of an Ellipse

The general equation of an ellipse centered at the origin is:

$$ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 $$

where:
- a is the length of the semi-major axis,
- b is the length of the semi-minor axis.

In our case, the given ellipse is:

$$ 9x^2 + 16y^2 = 144 $$

which can be rewritten in the standard form by dividing both sides by 144:

$$ \frac{9x^2}{144} + \frac{16y^2}{144} = 1 $$
$$ \frac{x^2}{16} + \frac{y^2}{9} = 1 $$

From this, we identify:
- $a^2 = 16 \Rightarrow a = 4$
- $b^2 = 9 \Rightarrow b = 3$

Thus, the ellipse has semi-major and semi-minor axes:
- $a = 4$ along the x-axis,
- $b = 3$ along the y-axis.

Note: Since $a > b$, the major axis is along the x-axis.

Ellipse in Standard Form

The equation now reads:

$$ \frac{x^2}{4^2} + \frac{y^2}{3^2} = 1 $$

which simplifies to:

$$ \frac{x^2}{16} + \frac{y^2}{9} = 1 $$

Determining the Arc of the First Quadrant

The first quadrant corresponds to the region where both x and y are positive. The arc of interest is the segment of the ellipse from the point $(a, 0)$ to $(0, b)$.

Finding Coordinates at the First Quadrant

- The endpoint on the x-axis: $(a, 0) = (4, 0)$
- The endpoint on the y-axis: $(0, b) = (0, 3)$

The arc length between these two points along the ellipse's perimeter defines the segment of the first quadrant.

Calculating the Arc Length of an Ellipse

Calculating the arc length of an ellipse is more complex than for a circle because the radius varies along the perimeter. The general formula for the arc length L from a point $\theta = 0$ to $\theta = \theta_1$ (parametrized in terms of the eccentric anomaly or parametric angle) is:

$$ L = \int_{0}^{\theta_1} \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \, d\theta $$

Alternatively, if parametrized explicitly:

$$
x = a \cos t \\
y = b \sin t
$$

then the differential arc length ds is:

$$
ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
$$

which simplifies to:

$$
ds = \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} \, dt

\]

Note: The parameter t runs from 0 to $\pi/2$ for the first quadrant.

Step-by-Step Calculation of the Arc Length

Let's now proceed to compute the arc length of the ellipse's first quadrant.

1. Parametrize the Ellipse

Using the parametric equations:

$$
x(t) = a \cos t = 4 \cos t \\
y(t) = b \sin t = 3 \sin t
$$

where:

$$
t \in [0, \pi/2]
$$

This parametrization traces the first quadrant from $(4, 0)$ to $(0, 3)$.

2. Write the Arc Length Integral

The length L of the arc from $t=0$ to $t=\pi/2$ is:

$$
L = \int_{0}^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
$$

Compute derivatives:

$$
\frac{dx}{dt} = -a \sin t = -4 \sin t \\
\frac{dy}{dt} = b \cos t = 3 \cos t
$$

So,

$$
ds = \sqrt{(-4 \sin t)^2 + (3 \cos t)^2} \, dt = \sqrt{16 \sin^2 t + 9 \cos^2 t} \, dt
$$

Therefore,

$$
L = \int_{0}^{\pi/2} \sqrt{16 \sin^2 t + 9 \cos^2 t} \, dt
$$

---

3. Simplify the Integral

Express the integrand:

$$
\sqrt{16 \sin^2 t + 9 \cos^2 t}
$$

which can be written as:

$$
\sqrt{(16 - 9) \sin^2 t + 9}
= \sqrt{7 \sin^2 t + 9}
$$

Thus, the integral becomes:

$$
L = \int_{0}^{\pi/2} \sqrt{7 \sin^2 t + 9} \, dt
$$

---

4. Recognize the Elliptic Integral

The integral:

$$
L = \int_{0}^{\pi/2} \sqrt{A \sin^2 t + B} \, dt
$$

is a form of elliptic integral, which generally does not have an elementary antiderivative.

In our case:

$$
A = 7, \quad B = 9
$$

The integral:

$$
L = \int_{0}^{\pi/2} \sqrt{7 \sin^2 t + 9} \, dt
$$

can be expressed in terms of the incomplete elliptic integral of the second kind, denoted as $E(\phi, k)$, or the complete elliptic integral $E(k)$ when $\phi = \pi/2$.

Expressing the Arc Length in Terms of Elliptic Integrals

Standard Form of Elliptic Integrals

The complete elliptic integral of the second kind is:

$$
E(k) = \int_{0}^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta} \, d\theta
$$

Our integral resembles this form but with different coefficients.

Rewriting the Integral

Factor out 9:

$$
L = \int_{0}^{\pi/2} \sqrt{9 + 7 \sin^2 t} \, dt = \int_{0}^{\pi/2} \sqrt{9 \left(1 + \frac{7}{9} \sin^2 t \right)} \, dt
$$

$$
L = 3 \int_{0}^{\pi/2} \sqrt{1 + \frac{7}{9} \sin^2 t} \, dt
$$

Note that:

$$
\sqrt{1 + m \sin^2 t} = \sqrt{1 + m \sin^2 t}
$$

which is similar to the form of elliptic integrals, but with the plus sign. To express the integral in standard elliptic form, we set:

$$
m = \frac{7}{9}
$$

and recognize that the integral:

$$
\int_{0}^{\pi/2} \sqrt{1 + m \sin^2 t} \, dt
$$

is known as the elliptic integral of the second kind $E(\phi | m)$.

Note: Since the form involves addition, the integral can be expressed as a combination of elliptic integrals, but it is most straightforward to evaluate numerically.
