How Large Should N Be To Guarantee That The Simpson's Rule Approximation To 1 9ex2 Dx 0 Is Accurate To

How Large Should N Be To Guarantee That The Simpson's Rule Approximation To $\int_0^1$ e^{x^2} dx Is Accurate To a desired degree of precision? Numerical integration is a fundamental aspect of applied mathematics, engineering, and computational sciences. Among the various techniques available, Simpson's rule stands out for its simplicity and relatively high accuracy when approximating definite integrals. However, the accuracy of Simpson's rule heavily depends on the number of subintervals, denoted as N. Understanding how to determine an appropriate N to achieve a specific accuracy is essential for efficient and reliable computations. This article explores the theoretical foundations, practical considerations, and step-by-step methods to determine the optimal N for Simpson's rule when approximating the integral of e^{x^2} from 0 to 1.

Introduction to Simpson's Rule and Its Significance

Simpson's rule is a numerical integration method that approximates the value of a definite integral by fitting quadratic polynomials through the function's values at specified points. Its formulation is derived from Simpson's 1/3 rule, which uses parabolic segments to estimate the area under a curve.

Why Use Simpson's Rule?

- High Accuracy for Smooth Functions: When the integrand is sufficiently smooth, Simpson's rule provides excellent approximations.
- Computational Efficiency: It requires relatively few function evaluations compared to other methods like Gaussian quadrature.
- Ease of Implementation: The formula is straightforward and easily programmable.

Basic Formula of Simpson's Rule

For an interval [a, b], subdivided into N even parts of width h = (b - a)/N, the approximation is:

\[

$$
\int_a^b f(x) dx \approx \frac{h}{3} \left[f(a) + 4 \sum_{i=1,3,5,...}^{N-1} f(a + i h) + 2 \sum_{i=2,4,6,...}^{N-2} f(a + i h) + f(b) \right]
$$

Understanding the Error in Simpson's Rule

The key to choosing N lies in understanding the error bound associated with Simpson's rule.

The Error Term

For a sufficiently smooth function f, the error E in Simpson's rule over [a, b] with N subintervals is given by:

$$
|E| \leq \frac{(b - a)^5}{180 N^4} \max_{x \in [a, b]} |f^{(4)}(x)|
$$

where $f^{(4)}(x)$ is the fourth derivative of f.

Implication of the Error Formula

- The error decreases rapidly as N increases because of the N^4 term in the denominator.
- To guarantee a specific accuracy ε, we need to estimate or bound the maximum of $|f^{(4)}(x)|$ over [a, b].

Applying Error Analysis to $\int_0^1 e^{x^2} dx$

The integral in question, $\int_0^1 e^{x^2} dx$, involves the function $f(x) = e^{x^2}$.

Calculating the Fourth Derivative $f^{(4)}(x)$

Determining the maximum of $|f^{(4)}(x)|$ over [0, 1] is essential for the error estimate.

- First derivative:

\[
f'(x) = 2x e^{x^2}
\]
- Second derivative:
\[
f''(x) = 2 e^{x^2} + 4x^2 e^{x^2} = 2 e^{x^2} (1 + 2 x^2)
\]
- Third derivative:
\[
f'''(x) = \frac{d}{dx} [2 e^{x^2} (1 + 2 x^2)]
\]
\[
= 2 \frac{d}{dx} [e^{x^2} (1 + 2 x^2)]
\]
Using product rule:
\[
= 2 [2 x e^{x^2} (1 + 2 x^2) + e^{x^2} (4 x)]
\]
\[
= 2 e^{x^2} [2 x (1 + 2 x^2) + 4 x]
\]
\[
= 2 e^{x^2} [2 x + 4 x^3 + 4 x]
\]
\[
= 2 e^{x^2} (6 x + 4 x^3)
\]
\[
= 2 e^{x^2} \cdot 2 x (3 + 2 x^2)
\]
\[
= 4 x e^{x^2} (3 + 2 x^2)
\]
- Fourth derivative:
\[
f^{(4)}(x) = \frac{d}{dx} [4 x e^{x^2} (3 + 2 x^2)]
\]
Apply product rule:
\[
= 4 \left[e^{x^2} (3 + 2 x^2) + x \frac{d}{dx} [e^{x^2} (3 + 2 x^2)]
\right]
\]

Calculate \(\frac{d}{dx} [e^{x^2} (3 + 2 x^2)]\):

\[
= 2 x e^{x^2} (3 + 2 x^2) + e^{x^2} (4 x)
\]
\[
= e^{x^2} [2 x (3 + 2 x^2) + 4 x]
\]

$$
= e^{x^2} [6 x + 4 x^3 + 4 x]
$$
$$
= e^{x^2} (10 x + 4 x^3)
$$

Putting it all together:

$$
f^{(4)}(x) = 4 \left[e^{x^2} (3 + 2 x^2) + x \cdot e^{x^2} (10 x + 4 x^3) \right]
$$
$$
= 4 e^{x^2} \left[(3 + 2 x^2) + x (10 x + 4 x^3) \right]
$$
$$
= 4 e^{x^2} \left[3 + 2 x^2 + 10 x^2 + 4 x^4 \right]
$$
$$
= 4 e^{x^2} \left[3 + 12 x^2 + 4 x^4 \right]
$$

Hence,

$$
f^{(4)}(x) = 4 e^{x^2} (3 + 12 x^2 + 4 x^4)
$$

Bounding $|f^{(4)}(x)|$ on [0, 1]

Since e^{x^2} is increasing on [0, 1], its maximum is at $x=1$:

$$
e^{1} = e \approx 2.71828
$$

Similarly, the polynomial $3 + 12 x^2 + 4 x^4$ is increasing for $x \ge 0$, reaching its maximum at $x=1$:

$$
3 + 12 (1) + 4 (1) = 3 + 12 + 4 = 19
$$

Therefore,

$$
|f^{(4)}(x)| \leq 4 \times e \times 19 \approx 4 \times 2.71828 \times 19 \approx 4 \times 2.71828 \times 19 \approx 4 \times 51.648 \approx 206.592
$$

Determining N for a Desired Accuracy

With the maximum of $|f^{(4)}(x)|$ estimated, we can now determine N.

Error Bound for Simpson's Rule

Given a target accuracy ε, the error bound:

$$
|E| \leq \frac{(b - a)^5}{180 N^4} \max_{x \in [a, b]} |f^{(4)}(x)|
$$

For the integral from 0 to 1:

$$
|E| \leq \frac{1^5}{180 N^4} \times 206.592 = \frac{206.592}{180 N^4}
$$

To ensure the approximation is within ε:

$$
\frac{206.592}{180 N^4} \leq \varepsilon
$$

$$
N^4 \geq \frac{206.592}{180 \varepsilon}
$$

\[
N \geq \left(\frac{206.

Frequently Asked Questions

How do I determine the minimum value of N needed for Simpson's Rule to approximate $\int_0^1 9e^{x^2}\, dx$ with a specified accuracy?

You use the error bound formula for Simpson's Rule: $|E| \leq (b - a)^5 / (180 N^4) \max|f^{(4)}(\xi)|$ for ξ in [a, b]. By estimating the maximum of the fourth derivative of the integrand, you can solve for N to meet your desired accuracy.

What is the importance of the fourth derivative in estimating the accuracy of Simpson's Rule?

The accuracy of Simpson's Rule depends on the magnitude of the fourth derivative of the function over the interval. A smaller maximum value of $|f^{(4)}(x)|$ leads to a smaller error bound, potentially allowing for a smaller N.

How can I estimate the maximum of the fourth derivative of $f(x) = 9e^{x^2}$ on [0,1]?

You can compute the fourth derivative symbolically or numerically and then find its maximum on [0,1]. Since the derivatives of e^{x^2} involve polynomial factors multiplied by e^{x^2}, their maximum occurs at the endpoint x=1, allowing for an estimate of the maximum.

Is there a straightforward formula to directly compute N for Simpson's Rule accuracy in this problem?

While a direct formula exists based on the error bound, it involves estimating the maximum of the fourth derivative. Once estimated, you substitute into the error formula and solve for N to ensure the desired accuracy.

If I want the Simpson's Rule approximation to be accurate within 10^{-4}, how do I determine N?

Estimate the maximum of $|f^{(4)}(x)|$ on [0,1], then use the error bound formula: $N \geq [((b - a)^5 / (180\ \varepsilon)) \max|f^{(4)}(\xi)|]^{1/4}$. Plug in $\varepsilon=10^{-4}$ to solve for N.

What are the practical steps to find the appropriate N for Simpson's Rule in this context?

First, compute or estimate the maximum of the fourth derivative of f(x). Second, decide on your desired error tolerance. Third, use the error bound formula to solve for N, ensuring the approximation meets the accuracy goal.

Can adaptive methods reduce the number of subintervals N needed for the same accuracy in Simpson's Rule?

Yes, adaptive Simpson's Rule adjusts the interval partitioning based on the function's behavior, often requiring fewer subintervals N overall to achieve the same accuracy compared to fixed N methods.

Are there other numerical integration methods that might require fewer subintervals than Simpson's Rule for the same accuracy?

Methods like Gaussian quadrature or adaptive quadrature can often achieve higher accuracy with fewer evaluation points or subintervals, especially for functions with smooth behavior, potentially reducing the required N.

Additional Resources

How Large Should N Be To Guarantee That The Simpson's Rule Approximation To $\int_0^1 e^{x^2} dx$ Is Accurate To?

Introduction

Numerical integration is an essential tool in mathematics, physics, engineering, and computer science, especially when dealing with functions that lack elementary antiderivatives or are computationally intensive to integrate analytically. Among various methods, Simpson's rule stands out due to its balance of simplicity and accuracy. However, the accuracy of Simpson's rule depends fundamentally on the number of subintervals, denoted by N, used to partition the integral's domain.

In this detailed discussion, we focus on the integral:

$$
I = \int_0^1 e^{x^2} \, dx
$$

and aim to determine how large N should be to ensure that the Simpson's rule approximation S_N achieves a specified accuracy, say, within a certain tolerance ε. This raises fundamental questions about error bounds, convergence rates, and practical implementation.

Understanding Simpson's Rule

Definition and Basic Properties

Simpson's rule approximates the definite integral of a function $f(x)$ over $[a, b]$ using quadratic interpolations over subintervals:

$$
S_N = \frac{\Delta x}{3} \left[f(x_0) + 4 \sum_{i=1,3,5,\dots}^{N-1} f(x_i) + 2 \sum_{i=2,4,6,\dots}^{N-2} f(x_i) + f(x_N) \right]
$$

where:

- N is an even number of subintervals,
- $\Delta x = \frac{b - a}{N}$,
- $x_i = a + i \Delta x$.

The approximation converges to the true integral as $N \to \infty$.

Error Bound of Simpson's Rule

The error estimate for Simpson's rule, assuming f is four times continuously differentiable on $[a, b]$, is given by:

$$
|E_S| \leq \frac{(b - a)^5}{180 N^4} \max_{x \in [a, b]} |f^{(4)}(x)|
$$

This formula highlights that the accuracy depends heavily on the fourth derivative of f. To determine N, we must analyze $f^{(4)}(x)$, estimate its maximum, and then solve for N.

Step 1: Analyzing the Function $f(x) = e^{x^2}$

The integral of interest:

$$
I = \int_0^1 e^{x^2} dx
$$

is non-elementary, but for error estimation, the function’s derivatives are crucial.

Derivatives of $f(x)$

- $f(x) = e^{x^2}$
- $f'(x) = 2x e^{x^2}$
- $f''(x) = 2 e^{x^2} + 4x^2 e^{x^2} = 2 e^{x^2} (1 + 2x^2)$
- $f'''(x) = 2 e^{x^2} (4x + 4x^3) = 8x e^{x^2} (1 + x^2)$
- $f^{(4)}(x) = \text{Derive } f'''(x)$

Calculating the fourth derivative:

$$
f^{(4)}(x) = \frac{d}{dx} \left[8x e^{x^2} (1 + x^2) \right]
$$

Applying the product rule:

\[

$$
f^{(4)}(x) = 8 \left[e^{x^2} (1 + x^2) + x \cdot \frac{d}{dx} \left(e^{x^2} (1 + x^2) \right) \right]
$$

Compute derivatives inside:

$$
\frac{d}{dx} \left(e^{x^2} (1 + x^2) \right) = e^{x^2} \cdot 2x (1 + x^2) + e^{x^2} \cdot 2x = e^{x^2} \left[2x (1 + x^2) + 2x \right]
$$

Simplify:

$$
e^{x^2} \left[2x + 2x^3 + 2x \right] = e^{x^2} \left(4x + 2x^3 \right)
$$

Putting everything together:

$$
f^{(4)}(x) = 8 \left[e^{x^2} (1 + x^2) + x \cdot e^{x^2} (4x + 2x^3) \right]
$$

$$
= 8 e^{x^2} \left[(1 + x^2) + x (4x + 2x^3) \right]
$$

$$
= 8 e^{x^2} \left[1 + x^2 + 4x^2 + 2x^4 \right]
$$

$$
= 8 e^{x^2} \left(1 + 5x^2 + 2x^4 \right)
$$

Step 2: Estimating $\max_{x \in [0,1]} |f^{(4)}(x)|$

Since e^{x^2} is increasing on $[0, 1]$, and the polynomial $1 + 5x^2 + 2x^4$ is non-decreasing on $[0, 1]$, the maximum occurs at $x=1$:

$$
|f^{(4)}(x)| \leq f^{(4)}(1) = 8 e^{1} (1 + 5 \cdot 1 + 2 \cdot 1) = 8 e \cdot (1 + 5 + 2) = 8 e \cdot 8 = 64 e
$$

Numerically, $e \approx 2.71828$):

$$
|f^{(4)}(x)| \leq 64 \times 2.71828 \approx 174.2
$$

\]

Thus,

$$
\max_{x \in [0,1]} |f^{(4)}(x)| \approx 174.2
$$

Step 3: Deriving the Required N

Applying the error bound:

$$
|E_S| \leq \frac{(b - a)^5}{180 N^4} \max_{x \in [a, b]} |f^{(4)}(x)|
$$

Set $a=0$, $b=1$, and define the desired accuracy ε:

$$
\varepsilon \geq \frac{1^5}{180 N^4} \times 174.2
$$

Simplify:

$$
\varepsilon \geq \frac{174.2}{180 N^4}
$$

Solve for N:

$$
N^4 \geq \frac{174.2}{180 \varepsilon}
$$

$$
N \geq \left(\frac{174.2}{180 \varepsilon} \right)^{1/4}
$$

Step 4: Practical Examples

Suppose we want the Simpson's rule approximation to be accurate within:

- $\varepsilon = 10^{-4}$,
- $\varepsilon = 10^{-6}$,
- $\varepsilon = 10^{-8}$.

Calculate N for each.

```
For \(\varepsilon = 10^{-4}\):

\[
N \geq \left( \frac{174.2}{180 \times 10^{-4}} \right)^{1/4} = \left(
\frac{174.2}{0.018} \right)^{1/4} \approx (9688.89)^{1/4}
\]

\[
N \approx \sqrt{\sqrt{9688.89}} \approx \sqrt{98.44} \approx 9.92
\]

Since \(N\) must be even:

\[
\boxed{N \geq 10}
\]

For \(\varepsilon = 10^{-6}\):

\[
N \geq \left( \frac{174.2}{180 \times 10^{-6}} \right)^{1/4} = \left(
\frac{174.2}{0.00018} \right)^{1/4} \approx (967777.78)^{1/4}
\]

Calculate:

\[
N \approx \sqrt{\sqrt{967777.78}} \approx \sqrt{984.7} \approx 31.36
\]

Round up to even:

\[
\boxed{N \geq 32}
\]

For \
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