If A And B Are Relatively Prime Positive Integers, Prove That The Diophantine Equation Ax - By = C Has

If A And B Are Relatively Prime Positive Integers, Prove That The Diophantine Equation Ax - By = C Has solutions under certain conditions, and explore the nature of these solutions, their existence, and their properties. This exploration is fundamental in number theory, particularly in understanding linear Diophantine equations, which are equations of the form Ax + By = C, where A, B, and C are integers, and solutions are sought in integers x and y. When A and B are relatively prime positive integers—that is, their greatest common divisor (gcd) is 1—the structure of solutions to the equation Ax - By = C exhibits interesting and significant properties.

This article aims to provide a comprehensive, well-organized analysis of this topic, including the necessary background, proof of solvability conditions, methods for finding solutions, and implications of these solutions. Whether you're a student learning number theory or a mathematics enthusiast seeking a deeper understanding, this detailed discussion will clarify the principles underlying the solvability of such Diophantine equations.

Understanding the Foundations of Diophantine Equations

What Are Diophantine Equations?

Diophantine equations are polynomial equations where solutions are sought in integers. Named after the ancient Greek mathematician Diophantus, these equations serve as a cornerstone of number theory. They take various forms, but the most classical and well-studied are linear Diophantine equations, which involve linear expressions.

Examples include:
- Ax + By = C
- Ax + By + Cz = D

In our context, the focus is on the equation:
$$ Ax - By = C $$
where A, B, and C are integers, and solutions are pairs (x, y) of integers satisfying the equation.

The Role of gcd in Solvability

A key concept in analyzing Diophantine equations is the greatest common divisor (gcd). For integers A and B, gcd(A, B) is the largest positive integer that divides both A and B without leaving a remainder.

- When A and B are relatively prime, gcd(A, B) = 1.
- The gcd influences whether solutions exist for equations like Ax + By = C, as per the fundamental theorem of linear Diophantine equations.

The Main Theorem: Solvability Conditions for Ax - By = C

Statement of the Theorem

Let A, B, and C be integers with A > 0 and B > 0, and suppose that A and B are relatively prime (i.e., gcd(A, B) = 1). The key theorem states:

The Diophantine equation $Ax - By = C$ has an integer solution (x, y) if and only if the gcd of A and B divides C.

Since gcd(A, B) = 1, the condition simplifies to:
$$1 \mid C$$
which is always true for any integer C.

Therefore, the equation $Ax - By = C$ has at least one integer solution for any integer C.

Implications of the Theorem

- When A and B are coprime positive integers, the linear Diophantine equation $Ax - By = C$ is always solvable regardless of the value of C.
- This contrasts with cases where A and B share a common divisor greater than 1, in which solutions exist only when that divisor divides C.

Proof of the Solvability Condition

Step 1: Establishing the Necessary Condition

Suppose (x_0, y_0) is an integer solution to:
$$Ax - By = C$$
then, for any solution, the following must hold:

- The left side, $Ax - By$, is divisible by gcd(A, B).
- Since A and B are coprime, gcd(A, B) = 1, so $1 \mid C$, which is always true.

Thus, the necessary condition for solutions is trivially satisfied when A and B are coprime.

Step 2: Showing the Sufficient Condition

Given that A and B are coprime, we want to show that for any integer C, solutions exist.

Method: Use the Extended Euclidean Algorithm

- The Extended Euclidean Algorithm allows us to find integers x_0 and y_0 such that:
$$A x_0 + B y_0 = 1$$
- Multiply both sides by C:
$$A x_0 C + B y_0 C = C$$
- Rearranged as:
$$A (x_0 C) + B (y_0 C) = C$$
- Set $x = x_0 C$ and $y = - y_0 C$, then:
$$A x - B y = C$$
- Since x_0 and y_0 are integers, x and y are integers, providing a particular solution.

Conclusion: For any integer C, there exists an integer solution (x, y) to the equation.

Constructing General Solutions

Particular Solution and General Form

Having established the existence of a particular solution (x_0, y_0), the general solution set can be described.

- The general solution to $Ax - By = C$ is given by:
$$x = x_0 + \frac{B}{\gcd(A, B)} t$$
$$y = y_0 + \frac{A}{\gcd(A, B)} t$$
where t is any integer.

- Since A and B are coprime, gcd(A, B) = 1, and the formulas simplify to:
$$x = x_0 + B t$$
$$y = y_0 + A t$$

Implication: Solutions form an infinite family parameterized by $t \in \mathbb{Z}$.

Example

Suppose A = 3, B = 4, and C = 5.

1. Find a particular solution:
- Using the Extended Euclidean Algorithm:
- $4 = 3 \times 1 + 1$
- $1 = 4 - 3 \times 1$
- From this, $1 = 4 - 3 \times 1$, so:
- $3 \times (-1) + 4 \times 1 = 1$
- Multiply both sides by C = 5:
- $3 \times (-5) + 4 \times 5 = 5$
- Particular solution:
- $x_0 = -5$, $y_0 = 5$

2. General solutions:
- $x = -5 + 4 t$
- $y = 5 + 3 t$

for any integer t.

Applications and Significance in Number Theory

Solving Linear Equations in Cryptography

In cryptography, many algorithms rely on solving linear Diophantine equations efficiently, especially in modular arithmetic contexts. The fact that coprime integers guarantee solutions informs key generation, encryption, and decryption processes.

Integer Programming and Optimization

Understanding the solutions to equations like $Ax - By = C$ is crucial in integer programming, where variables are constrained to be integers, and solutions must satisfy linear equations.

Number Theoretic Functions and Factorization

The properties of coprimality and the structure of solutions aid in the study of number-theoretic functions such as Euler's totient function, and in algorithms for factorization and primality testing.

Limitations and Extensions

When A and B Are Not Coprime

If gcd(A, B) = d > 1, then solutions exist only when $d \mid C$. In such cases:
- Divide the entire equation by d:
$$\frac{A}{d} x - \frac{B}{d} y = \frac{C}{d}$$
- The resulting equation has solutions if and only if $d \mid C$.

Non-Linear and Higher-Degree Equations

While the current discussion focuses on linear equations, similar principles extend in more complex polynomial equations, but the analysis becomes significantly more involved.

Conclusion

In summary, when A and B are relatively prime positive integers, the Diophantine equation $Ax - By = C$ is always solvable for any integer C. This stems from the fundamental properties of coprime integers and the Euclidean Algorithm, which guarantees the existence of integer solutions. The solutions form an infinite family, characterized by a parameter t, reflecting the rich structure of linear Diophantine equations.

This understanding is vital across various domains in mathematics and computer science, including cryptography, algorithm design, and number theory research. Recognizing the condition that gcd(A, B) divides C as a necessary and sufficient criterion for solutions underpins many practical applications and theoretical advancements.

By mastering these principles, students and

Frequently Asked Questions

What is the necessary and sufficient condition for the Diophantine equation Ax - By = C to have integer solutions when A and B are relatively prime positive integers?

The equation has integer solutions if and only if the greatest common divisor of A and B, which is 1, divides C. Since A and B are coprime, solutions exist

for any integer C.

How does the coprimality of A and B influence the solvability of the equation Ax - By = C?

Because A and B are coprime, their linear combination can produce any integer C, ensuring solutions exist for all C divisible by 1, i.e., for all integers C.

What is the general form of solutions to the equation Ax - By = C when A and B are coprime positive integers?

The general solutions are given by $x = x_0 + (B/d)t$ and $y = y_0 + (A/d)t$, where (x_0, y_0) is a particular solution to the equation, d=1, and t is any integer.

Can the equation Ax - By = C have no solutions if A and B are coprime? Why or why not?

No, because A and B are coprime, they satisfy gcd(A, B) = 1, which divides any integer C, ensuring solutions exist for all C.

What role does Bezout’s Identity play in solving the Diophantine equation Ax - By = C?

Bezout’s Identity states that since gcd(A, B) = 1, there exist integers x and y such that Ax + By = 1. Scaling these solutions gives solutions to Ax - By = C.

How many solutions does the equation Ax - By = C have when A and B are coprime positive integers?

There are infinitely many solutions because, once a particular solution is found, the general solution involves adding integer multiples of (B, A) to generate an infinite set.

If A and B are coprime, what are the conditions on C for the equation Ax - By = C to have solutions?

Since gcd(A, B) = 1 divides every integer, the equation has solutions for all integer values of C.

How does the Euclidean Algorithm assist in solving

the equation Ax - By = C for coprime A and B?

The Euclidean Algorithm helps find explicit particular solutions (x_0, y_0) by computing the coefficients in Bezout’s Identity, which then generate the general solutions.

What is the significance of the parameter t in the general solution to Ax - By = C?

The parameter t accounts for the infinitely many solutions, representing shifts along the solution space by multiples of (B, A).

Are solutions to Ax - By = C always integers when A and B are coprime? Under what conditions?

Yes, solutions are always integers for any integer C because the coprimality ensures that the linear combination can produce any integer C.

Additional Resources

Understanding the Diophantine Equation $A x - B y = C$ When $\gcd(A, B) = 1$

The exploration of linear Diophantine equations is a foundational aspect of number theory, with applications spanning cryptography, algorithm design, and algebraic number theory. In particular, the equation:

$$
A x - B y = C
$$

where A, B, C are integers, and seeking integer solutions (x, y), offers a rich ground for analysis. When the positive integers A and B are relatively prime, i.e., their greatest common divisor $\gcd(A, B) = 1$, this simplifies the solution structure considerably. This review delves into the conditions under which solutions exist, the nature and number of solutions, and the methods to explicitly find them.

Fundamentals of Linear Diophantine Equations

Definition and Basic Properties

A linear Diophantine equation is an equation of the form:

$$
A x + B y = C
$$

where A, B, C are integers, and solutions are sought in integers x, y. The equation

$$
A x - B y = C
$$

is a special case, differing only by a sign, and shares the same fundamental properties.

Key points:

- Existence of solutions: Solutions exist if and only if $\gcd(A, B)$ divides C.

- General solution form: When solutions exist, they form an infinite set generated by a particular solution and the homogeneous solutions.

- Number of solutions: If solutions exist, there are infinitely many, parametrized by an integer parameter.

Role of $\gcd(A, B) = 1$: Simplification and Implications

Why is $\gcd(A, B) = 1$ significant?

When A and B are relatively prime, the key implications are:

- Existence of solutions for any integer C: Because $\gcd(A, B) = 1$, it divides every integer C, so solutions exist for all $C \in \mathbb{Z}$.

- Simpler solution structure: The solutions can be explicitly constructed using the Extended Euclidean Algorithm, which guarantees the existence of integer solutions for such cases.

- Uniqueness considerations: The solutions are more straightforward to describe; in particular, the structure of the solution set becomes more

transparent.

In essence:

$$
\text{If } \gcd(A, B) = 1, \text{ then } \text{the equation } A x - B y = C \text{ has solutions for all } C \in \mathbb{Z}.
$$

Existence of Solutions: Necessary and Sufficient Conditions

The Fundamental Theorem

For the linear Diophantine equation:

$$
A x - B y = C,
$$

the key theorem states:

> The equation has integer solutions if and only if $\gcd(A, B)$ divides C.

Given that $\gcd(A, B) = 1$, this condition simplifies dramatically.

Application When $\gcd(A, B) = 1$

Since 1 divides every integer C, the necessary and sufficient condition reduces to:

$$
\boxed{
\text{For all } C \in \mathbb{Z}, \text{ the equation } A x - B y = C \text{ has solutions}.
}
$$

This means:

- For any integer C, solutions (x, y) exist.

- The problem of solving the equation becomes straightforward, provided we can find one particular solution.

Summary:

Condition	Implication
$\gcd(A, B) = 1$	Equation has solutions for all $C \in \mathbb{Z}$.

Finding Solutions: The Role of the Extended Euclidean Algorithm

Constructing a Particular Solution

When $\gcd(A, B) = 1$, the Extended Euclidean Algorithm provides a systematic way to find integers x_0, y_0 satisfying:

$$
A x_0 - B y_0 = 1.
$$

Once such a particular solution to the homogeneous equation:

$$
A x - B y = 1,
$$

is obtained, solutions to the inhomogeneous equation:

$$
A x - B y = C,
$$

can be derived by scaling.

Step-by-step process:

1. Apply Extended Euclidean Algorithm:

- Find integers x_1, y_1 such that:

$$
A x_1 + B y_1 = 1.
$$

(Note: The original equation has a minus sign, but since $\gcd(A, B) = 1$, the signs can be adjusted accordingly.)

2. Obtain a particular solution:

- Multiply both sides by C:

$$
A (C x_1) + B (C y_1) = C.
$$

- Rearranged as:

$$
A (x_0) - B (y_0) = C,
$$

where:

$$
x_0 = C x_1, \quad y_0 = - C y_1,
$$

or adjusted depending on the sign conventions.

3. General solution:

- The set of all solutions is then:

$$
x = x_0 + k B,
$$
$$
y = y_0 + k A,
$$

where $k \in \mathbb{Z}$.

Explicit Solution Formula

Based on the above, the explicit formulas are:

$$
\boxed{
\begin{aligned}
x &= x_0 + k B, \\
y &= y_0 + k A,
```
```

\end{aligned}
}
\]

where:

- \(x_0, y_0\) is a particular solution to \(A x - B y = C\),
- \(k\) is any integer.

This parametrization describes an infinite family of solutions, with each integer \(k\) producing a different solution.

Solution Characterization and Number of Solutions

Infinite Solutions

As shown, when \(\gcd(A, B)=1\), solutions exist for every \(C\), and they form an infinite set parametrized by \(k\).

Solutions with Additional Constraints

Often, practical problems impose constraints such as:

- \(x, y \geq 0\),
- or within certain bounds.

Under such constraints, the number of solutions becomes finite, and determining it involves analyzing the inequalities:

\[
x_0 + k B \geq 0,
\]
\[
y_0 + k A \geq 0,
\]

and solving for the integer \(k\).

Special Cases and Variations

When $C = 0$

The homogeneous equation:

$$
A x - B y = 0,
$$

has solutions:

$$
x = m B, \quad y = m A, \quad m \in \mathbb{Z}.
$$

This forms a lattice of solutions along a line through the origin.

When A and B are positive

The positivity of A, B simplifies the interpretation of solutions, especially when considering constraints like non-negativity, which are common in practical applications.

Applying to Other Variants

The techniques and results extend to equations with different signs and coefficients, with careful attention to the divisibility conditions and the structure of solutions.

Summary and Conclusions

- Existence of solutions: When $\gcd(A, B) = 1$, the Diophantine equation $A x - B y = C$ has solutions for every integer C.

- Method to find solutions:

1. Use the Extended Euclidean Algorithm to find particular solutions to $A x + B y = 1$.

2. Scale these solutions by C to obtain particular solutions to $A x - B y = C$.

3. Generate the entire solution set by adding multiples of (B, A).

- Number of solutions: Infinite, parametrized by an integer k.

- Practical considerations: For constraints like non-negativity, the solution set must be intersected with the relevant inequalities, which can often be handled systematically once the general solution is known.

Final Remarks

The case where $\gcd(A, B) = 1$ provides a clean and elegant scenario in the study of linear Diophantine equations, exemplifying the power of the Euclidean

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