If I :=[a;b] And I' = [a'; B'] Are Closed Intervals In R, Show That I I' If And Only If A' A And B B'.

If I := [a; b] And I' = [a'; b'] Are Closed Intervals In R, Show That I ⊆ I' If And Only If a' ≤ a And b ≤ b'.

Introduction

In the real number system, closed intervals are fundamental objects that capture the idea of all points lying between two bounds, inclusive. Understanding the inclusion relationship between two closed intervals is crucial in various fields such as analysis, topology, and applied mathematics. The problem at hand is to rigorously demonstrate that one closed interval is contained within another if and only if the lower bound of the first is greater than or equal to the lower bound of the second, and the upper bound of the first is less than or equal to the upper bound of the second.

This statement formalizes the intuitive idea: for the interval [a; b] to be a subset of [a'; b'], every point in [a; b] must also lie in [a'; b']. We will explore this equivalence thoroughly, providing formal proof and discussing its implications.

Definitions and Notation

Before diving into the proof, it is essential to clarify the notations and definitions used:

Closed Intervals in R

- A closed interval [a; b], with real numbers a and b, is the set:
$$
[a; b] = \{ x \in \mathbb{R} : a \leq x \leq b \}
$$
- Similarly, another closed interval [a'; b'] is defined as:
$$
[a'; b'] = \{ x \in \mathbb{R} : a' \leq x \leq b' \}
$$

Subset Relation

- The notation $I \subseteq I'$ (or $I \subset I'$) indicates that every point in interval I is also in I'.

Statement of the Theorem

Theorem:
Given two closed intervals $I = [a; b]$ and $I' = [a'; b']$, the following are equivalent:

$$
I \subseteq I' \quad \Longleftrightarrow \quad a' \leq a \quad \text{and} \quad b \leq b'
$$

Proof of the Theorem

The proof proceeds in two parts: showing the forward implication ($\Rightarrow$) and the reverse implication ($\Leftarrow$).

Part 1: If $I \subseteq I'$, then $a' \leq a$ and $b \leq b'$

Assumption:
Suppose $I = [a; b] \subseteq [a'; b'] = I'$.

Goal:
Prove that $a' \leq a$ and $b \leq b'$.

Proof:

1. Since $a \in I$, and $I \subseteq I'$, then $a \in I'$.
- By the definition of I', this implies:
$$
a' \leq a \leq b' \quad \Rightarrow \quad a' \leq a
$$

2. Similarly, since $b \in I$, and $I \subseteq I'$, then $b \in I'$.
- Therefore:
\[

a' \leq b \leq b' \quad \Rightarrow \quad b \leq b'
\]

Conclusion:
From the above, we have established:
\[
a' \leq a \quad \text{and} \quad b \leq b'
\]

Part 2: If \(a' \leq a \) and \(b \leq b' \), then \(I \subseteq I' \)

Assumption:
Suppose:
\[
a' \leq a \quad \text{and} \quad b \leq b'
\]

Goal:
Show that \([a; b] \subseteq [a'; b'] \).

Proof:

- Take an arbitrary \(x \in [a; b] \).
- By the definition of the interval, \(a \leq x \leq b \).

- Since \(a' \leq a \), it follows that:
\[
a' \leq a \leq x
\]
Hence,
\[
a' \leq x
\]

- Similarly, since \(x \leq b \) and \(b \leq b' \), we get:
\[
x \leq b \leq b'
\]
Thus,
\[
x \leq b'
\]

- Combining these inequalities:
\[
a' \leq x \leq b'
\]

- Therefore, $x \in [a'; b']$.

- Because x was an arbitrary element of $[a; b]$, we conclude:

$$
[a; b] \subseteq [a'; b']
$$

Summary and Implications

The proof confirms the intuitive understanding of interval inclusion: a closed interval $[a; b]$ is contained within $[a'; b']$ precisely when the lower bound of the first is greater than or equal to the lower bound of the second, and the upper bound of the first is less than or equal to the upper bound of the second.

Implications:

- This characterization allows for quick verification of subset relations between intervals.
- It is foundational in analysis, especially in proofs involving convergence, limits, and continuity.
- The result extends naturally to other types of intervals (open, half-open), with similar inequalities but adjusted definitions.

Additional Remarks

- This theorem exemplifies how order relations in $\mathbb{R}$ translate directly into set inclusion relationships between intervals.
- The proof relies fundamentally on the properties of real numbers, particularly the order relation $\leq$.
- The result remains valid whether the bounds are finite or infinite (e.g., $[a; \infty)$), with appropriate modifications.

Conclusion

In conclusion, we have rigorously demonstrated that for two closed intervals $I = [a; b]$ and $I' = [a'; b']$, the subset relation $I \subseteq I'$ holds if and only if $a' \leq a$ and $b \leq b'$. This equivalence encapsulates the intuitive notion of one interval lying entirely within another and provides a precise criterion for inclusion based solely on their endpoints.

Understanding this relationship is fundamental in the study of real analysis, forming the basis for more complex concepts such as intervals' ordering, topology of \(\mathbb{R} \), and the behavior of functions defined over intervals.

Frequently Asked Questions

What does the notation $I := [a; b]$ and $I' := [a'; b']$ represent in the context of closed intervals in $\mathbb{R}$?

The notation $I := [a; b]$ and $I' := [a'; b']$ represents closed intervals in $\mathbb{R}$, where each interval includes its endpoints. Specifically, $I = [a, b]$ contains all real numbers x such that $a \leq x \leq b$, and similarly for $I' = [a', b']$.

What is the meaning of the statement '$I \subseteq I'$' in the context of closed intervals?

The statement '$I \subseteq I'$' means that the interval I is a subset of I', i.e., every element of I is also an element of I'. This translates to all elements between a and b being contained within $[a', b']$.

How can we interpret the condition '$I \subseteq I'$' in terms of the endpoints a, b, a', and b'?

For $I = [a, b]$ to be a subset of $I' = [a', b']$, it must be that $a' \leq a$ and $b \leq b'$. This ensures the entire interval $[a, b]$ lies within $[a', b']$.

What is the statement we need to prove regarding $I \subseteq I'$ and the endpoints?

We need to show that $I \subseteq I'$ if and only if $a' \leq a$ and $b \leq b'$. In other words, the inclusion of intervals is equivalent to the inequalities relating their endpoints.

How do you prove the 'if' part of the statement: if $a' \leq a$ and $b \leq b'$, then $I \subseteq I'$?

Assuming $a' \leq a$ and $b \leq b'$, take any $x \in I$. Since $a \leq x \leq b$ and $a' \leq a$, $b \leq b'$, it follows that $a' \leq x \leq b'$, so $x \in I'$. Therefore, every element of I is in I', proving $I \subseteq I'$.

How do you prove the 'only if' part of the statement: if $I \subseteq I'$, then $a' \leq a$ and $b \leq b'$?

Suppose $I \subseteq I'$. Since $a \in I$, it must be in I', so $a \in [a', b']$. Therefore, $a' \leq a$. Similarly, since $b \in I$, it must be in I', so $b \in [a', b']$, which implies $b \leq b'$. Hence, $a' \leq a$ and $b \leq b'$.

What is the conclusion of the proof regarding the characterization of interval inclusion?

The proof concludes that the closed interval $[a, b]$ is a subset of $[a', b']$ if and only if the endpoints satisfy $a' \leq a$ and $b \leq b'$. This provides a simple criterion for interval inclusion based on endpoint inequalities.

Additional Resources

Understanding the Conditions for Intersection of Closed Intervals in Real Numbers: An In-Depth Analysis

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Introduction

The concept of intervals is fundamental in real analysis, serving as the building blocks for understanding continuity, limits, and various other properties of real-valued functions. In particular, closed intervals in the real number line, denoted as $[a; b]$, are crucial because they contain all their boundary points, providing a complete and 'closed' set which is often easier to analyze in mathematical contexts.

A common question encountered when working with intervals is determining whether two such sets intersect, i.e., whether their intersection is non-empty. This seemingly simple question turns out to have a precise and elegant answer when dealing with closed intervals: Two closed intervals intersect if and only if the endpoint of one overlaps with the other, specifically, if the left endpoint of the second interval is less than or equal to the right endpoint of the first, and vice versa.

In this article, we explore this statement in detail. We analyze the condition that if $I = [a; b]$ and $I' = [a'; b']$ are closed intervals in $\mathbb{R}$, then $I \cap I' \neq \emptyset$ if and only if $a' \leq b$ and $a \leq b'$. We will systematically dissect this statement, provide rigorous proofs, interpret the geometric intuition, and discuss its implications.

Background and Definitions

Closed Intervals in $\mathbb{R}$

A closed interval $[a; b]$ in $\mathbb{R}$ is defined as the set:

$$
[a; b] = \{ x \in \mathbb{R} : a \leq x \leq b \}
$$

where $a, b \in \mathbb{R}$ and $a \leq b$. Such an interval contains all points between a and b, inclusive. These intervals are fundamental in analysis because:

- They are compact sets.
- They are closed and bounded.
- They serve as domains for continuous functions in many theorems.

Intersection of Intervals

Given two sets $A, B \subseteq \mathbb{R}$, their intersection is:

$$
A \cap B = \{ x \in \mathbb{R} : x \in A \text{ and } x \in B \}
$$

In the context of intervals, the intersection of two closed intervals is either another interval (possibly degenerate to a point or empty set) or empty.

The Main Theorem: Conditions for Non-Empty Intersection

Statement:

Given two closed intervals in $\mathbb{R}$:

$$
I = [a; b], \quad I' = [a'; b']
$$

then,

$$
I \cap I' \neq \emptyset \quad \text{if and only if} \quad a' \leq b \quad \text{and} \quad a \leq b'
$$

Equivalently, the intervals intersect if and only if the intervals are not disjoint, i.e., one does not lie entirely to the left or right of the other. The inequalities capture this precisely.

Detailed Explanation of the Conditions

The "If" Part: Showing that if $a' \leq b$ and $a \leq b'$, then $I \cap I' \neq \emptyset$

Intuitive reasoning:

- If $a' \leq b$, then the start of I' is at or to the left of the end of I.
- If $a \leq b'$, then the start of I is at or to the left of the end of I'.
- These conditions together imply that the intervals overlap or at least touch at some point.

Formal proof:

Suppose:

$$
a' \leq b \quad \text{and} \quad a \leq b'
$$

Consider the potential intersection point:

$$
x = \max(a, a')
$$

Since both a and a' are less than or equal to the other's endpoint, the following holds:

$$
x \geq a, \quad x \geq a'
$$

and

$$
x \leq b, \quad x \leq b'
$$

because:

- $a \leq b'$ implies $a \leq b'$,
- $a' \leq b$ implies $a' \leq b$.

Therefore, x lies in both intervals:

$$

x \in [a; b] \quad \text{and} \quad x \in [a'; b']
\]

which means:

\[
x \in I \cap I'
\]

and hence the intersection is non-empty.

The "Only If" Part: Showing that if \(I \cap I' \neq \emptyset\), then \(a' \leq b\) and \(a \leq b'\)

Intuitive reasoning:

- If the intervals share at least one point, then their start and end points must satisfy the inequalities.
- Otherwise, the intervals would be disjoint.

Formal proof:

Suppose:

\[
I \cap I' \neq \emptyset
\]

then, there exists \(x \in \mathbb{R}\) such that:

\[
x \in [a; b] \quad \text{and} \quad x \in [a'; b']
\]

which implies:

\[
a \leq x \leq b, \quad a' \leq x \leq b'
\]

From these inequalities:

- Since \(x \geq a\) and \(x \leq b'\), it follows that:

\[
a \leq x \leq b' \Rightarrow a \leq b'
\]

- Similarly, since \(x \geq a'\) and \(x \leq b\), then:

\[

$$
a' \leq x \leq b \Rightarrow a' \leq b
$$

Thus, the necessary inequalities for the intersection to be non-empty are:

$$
a' \leq b \quad \text{and} \quad a \leq b'
$$

Geometric Interpretation

The inequalities describe the relative positions of the two intervals on the real line:

- If $a' \leq b$ and $a \leq b'$, the intervals overlap or touch. Geometrically, the right endpoint of the leftmost interval is not less than the left endpoint of the other, ensuring intersection.

- If either inequality fails, the intervals are disjoint, with one lying entirely to the left or right of the other.

Visual illustration:

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Interval I: [a------------b]
Interval I': [a'--------b']
```

- When $a' \leq b$ and $a \leq b'$, these intervals overlap or meet at a point.
- When $a' > b$, I' is entirely to the right of I.
- When $a > b'$, I is entirely to the right of I'.

Special Cases and Degenerate Intervals

When $a = b$ or $a' = b'$:

- The intervals reduce to points.
- The intersection is either a point (if the points coincide) or empty.

When the intervals are degenerate, the same conditions hold:

$$
[a; a], \quad [a'; a']
$$

and intersection occurs precisely when the points are equal:

$$
a = a'
$$

which satisfies the inequalities:

$$
a' \leq a, \quad a \leq a'
$$

i.e., both inequalities hold iff $a = a'$.

Practical Implications and Applications

1. Interval Scheduling and Overlaps

In computational contexts like scheduling, understanding whether two time intervals overlap is essential. The derived condition provides an efficient check:

- Given two time intervals, verify whether $a' \leq b$ and $a \leq b'$.

2. Real Analysis and Measure Theory

The intersection condition is fundamental when analyzing properties like compactness, continuity, and measure. It helps in:

- Partitioning sets.
- Constructing partitions of unity.
- Understanding the behavior of functions over overlapping domains.

3. Numerical Algorithms

Efficient algorithms for detecting overlaps in large datasets (e.g., genomic intervals, event timelines) rely on such inequalities for quick pruning.

Summary and Conclusion

The analysis of the intersection of closed intervals in $\mathbb{R}$ reveals a simple yet powerful criterion: Two closed intervals $[a; b]$ and $[a'; b']$ intersect if and only if $a' \leq b$ and $a \leq b'$.

This criterion hinges on the fundamental idea that for two sets to share at least one point, their ranges must overlap or meet at a boundary. The proof involves elementary inequalities and the properties of real numbers, making it an elegant demonstration of the interplay between algebra and geometry.

Understanding this condition is more than an academic exercise

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