In A Space Where Z>=0, Find The Mass Of The Crystal Massbelow X^2+y^2+z^2=4 And Above Z=0, And Find

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Understanding how to calculate the mass of a solid region in three-dimensional space is fundamental in multivariable calculus, especially when dealing with volume integrals. In this particular problem, we are asked to find the mass of a crystal-shaped region bounded above by the sphere $x^2 + y^2 + z^2 = 4$ and below by the plane $z=0$, within the space where $z \geq 0$. This description suggests a hemispherical cap, often referred to as the upper hemisphere, with a radius of 2.

This problem combines multiple concepts: setting up the appropriate coordinate system, defining the bounds of integration, and executing the volume integral considering the density function—if any is specified. Often, in such problems, the density is assumed uniform unless otherwise specified, simplifying the calculation to finding the volume of the region, which then serves as the mass.

In this article, we will explore the problem step-by-step, covering the necessary background, choosing suitable coordinate systems, setting up the integral, and computing the volume—or mass—of the region. This comprehensive overview aims to help you understand similar problems in multivariable calculus involving volume calculations in spherical regions.

Understanding the Geometry of the Region

Before setting up the integral, it is crucial to understand the geometric shape and boundaries:

The Sphere $x^2 + y^2 + z^2 = 4$

- This is a sphere centered at the origin with radius 2.
- The equation describes all points in space exactly 2 units from the origin.

The Plane $z=0$

- This plane divides the sphere into two hemispheres.
- Since the problem specifies $z \geq 0$, we are concerned with the upper hemisphere.

The Region of Interest

- The region is bounded above by the sphere and below by the plane $z=0$.
- The region is the upper half of the sphere, forming a hemispherical cap.

Visualizing this, imagine a sphere of radius 2 sitting above the xy-plane, with the boundary at $z=0$.

Choosing the Coordinate System

Given the symmetry of the problem, spherical coordinates are the most natural choice for this calculation. Spherical coordinates (ρ, θ, ϕ) are related to Cartesian coordinates (x, y, z) by:

$$
\begin{cases}
x = \rho \sin \phi \cos \theta \\
y = \rho \sin \phi \sin \theta \\
z = \rho \cos \phi
\end{cases}
$$

where:
- $\rho \geq 0$ is the distance from the origin,
- $0 \leq \phi \leq \pi$ is the angle from the positive z-axis,
- $0 \leq \theta < 2\pi$ is the azimuthal angle in the xy-plane.

In our case:
- The boundary of the sphere: $\rho = 2$,
- The boundary $z=0$ corresponds to $\phi = \pi/2$ (since $z = \rho \cos \phi$), with $z \geq 0$ corresponding to $0 \leq \phi \leq \pi/2$.

Thus, the region can be described as:
- $0 \leq \theta \leq 2\pi$,
- $0 \leq \rho \leq 2$,
- $0 \leq \phi \leq \pi/2$.

Setting Up the Volume Integral

The mass M of the region (assuming uniform density, which is typically 1 unless specified otherwise) can be calculated via a triple integral:

$$
M = \iiint_{V} \rho \, dV
$$

In spherical coordinates, the volume element dV is:

$$
dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
$$

Since the density is uniform and assumed to be 1, the mass simplifies to the volume:

$$
M = \int_{0}^{2\pi} \int_{0}^{\pi/2} \int_{0}^{2} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
$$

Calculating the Volume (Mass)

Let's perform the integration step-by-step.

Integrate with respect to ρ:

$$
\int_{0}^{2} \rho^2 \, d\rho = \left[\frac{\rho^3}{3} \right]_0^2 = \frac{8}{3}
$$

Integrate with respect to ϕ:

$$
\int_{0}^{\pi/2} \sin \phi \, d\phi = \left[-\cos \phi \right]_0^{\pi/2} = -\cos (\pi/2) + \cos (0) = 0 + 1 = 1
$$

Integrate with respect to θ:

$$
\int_{0}^{2\pi} d\theta = 2\pi
$$

Putting it all together:

$$
M = \left(\frac{8}{3}\right) \times 1 \times 2\pi = \frac{16\pi}{3}
$$

Thus, the mass of the crystal, which coincides with the volume of the upper hemisphere with radius 2, is:

$$
\boxed{
M = \frac{16\pi}{3}
}
$$

Summary of Key Steps and Concepts

- Identified the geometric region: the upper hemisphere of radius 2, above the xy-plane.
- Chose the appropriate coordinate system: spherical coordinates for symmetry.
- Set the bounds:
- $\theta: 0 \to 2\pi$,
- $\phi: 0 \to \pi/2$,
- $\rho: 0 \to 2$.
- Expressed the volume element: $dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$.
- Performed the integrations sequentially: with respect to ρ, ϕ, then θ.
- Obtained the volume (mass): $\frac{16\pi}{3}$.

Applications and Additional Considerations

This problem demonstrates a typical application of triple integrals in calculating the volume and mass of symmetrical regions. Such calculations are fundamental in physics and engineering, especially in:

- Material sciences: determining mass distribution in objects.
- Astrophysics: modeling celestial bodies.
- Fluid dynamics: computing the volume of fluid within boundaries.

Variations of the Problem

Depending on the problem, you might encounter variations such as:

- Non-uniform density functions $\rho(x, y, z)$,
- Different boundary surfaces or planes,
- Regions bounded by other spheres or complex surfaces.

In each case, the general approach remains similar:

1. Understand the geometry.
2. Choose the most suitable coordinate system.
3. Set up the bounds based on the geometry.
4. Write the integral with the appropriate volume element.
5. Integrate step-by-step.

Conclusion

Calculating the mass of a solid region in space requires a careful understanding of the geometry and appropriate choice of coordinate systems. For the problem involving the upper hemisphere of radius 2, spherical coordinates simplify the integration process significantly. The resulting volume, and hence the mass for uniform density, is $\frac{16\pi}{3}$.

Mastery of these techniques enables solving a wide range of problems involving three-dimensional regions bounded by spheres, planes, and other surfaces.

Whether you're a student learning multivariable calculus or a professional applying these concepts, understanding the process of setting up and evaluating such integrals is essential. With practice, recognizing the symmetry and choosing the right coordinate system becomes intuitive, making complex volume calculations manageable and straightforward.

Frequently Asked Questions

What is the geometric region described in the problem involving the sphere and the plane?

The region is the portion of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the plane $z = 0$, i.e., the upper hemisphere including its boundary, in the space where $z \geq 0$.

How do you set up the integral to find the mass of the crystal in this region?

Assuming uniform density, the mass is obtained by integrating the density over the volume. If the density is uniform, the mass equals the volume of the region, so set up a triple integral in spherical coordinates or cylindrical coordinates over the specified region.

What coordinate system is most suitable for calculating the volume of a spherical cap above z=0?

Spherical coordinates are most suitable because the region is part of a sphere, simplifying the limits of integration based on angles and radius.

What are the limits of integration for the spherical coordinates in this problem?

In spherical coordinates (r, θ, φ), with r from 0 to 2 (radius of the sphere), θ from 0 to $\pi/2$ (since $z \geq 0$), and φ from 0 to 2π, the limits cover the upper hemisphere.

How do you compute the volume of the spherical cap above z=0 within the sphere?

The volume of the upper hemisphere ($z \geq 0$) of a sphere with radius 2 is $(1/2)(4/3)\pi\ 2^3 = (1/2)\ (4/3)\pi\ 8 = (4/3)\pi\ 4 = (16/3)\pi$.

If the density of the crystal is uniform, how does that affect the calculation of its mass?

The mass is equal to the density multiplied by the volume of the region. With uniform density, the problem reduces to calculating the volume of the upper hemisphere, which is (16/3)π.

What is the final answer for the mass of the crystal if the density is given as ρ?

The mass is M = ρ × (16/3)π, where ρ is the constant density of the crystal.

Additional Resources

In A Space Where Z ≥ 0, Find The Mass Of The Crystal Mass Below $X^2 + Y^2 + Z^2 = 4$ And Above Z = 0, And Find

Understanding the problem of calculating the mass of a crystal-shaped object within a specific region is a fundamental task in multivariable calculus, especially when dealing with volumes bounded by surfaces. The statement suggests a solid region in three-dimensional space constrained between the upper surface of a sphere of radius 2 and the plane z=0 (the XY-plane). This region resembles a hemispherical cap or a solid hemisphere. The challenge involves integrating a density function over this region to find the total mass, which can be generalized for various density distributions.

In this review, we will explore the problem in depth—breaking down the geometry, setting up the integrals, solving step-by-step, and discussing potential variations and complexities. The goal is to provide a comprehensive guide suitable for students, educators, or anyone interested in volume integrals in three dimensions.

Understanding the Geometric Region

1. The Sphere and Its Equations

The sphere given by the equation:

$$ x^2 + y^2 + z^2 = 4 $$

has a radius of 2, centered at the origin $(0,0,0)$. The sphere's surface encloses a volume of interest, which is a hemisphere when considering only $z \geq 0$.

Key points:
- The sphere extends from -2 to 2 along the $x, y,$ and z axes.
- The upper hemisphere corresponds to $z \geq 0$.

2. The Region of Interest

Since the problem specifies "below $x^2 + y^2 + z^2 = 4$ and above $z=0$", the region is:

- The upper half of the sphere $x^2 + y^2 + z^2 \leq 4$,
- constrained by the plane $z=0$ (the XY-plane).

This region is a hemispherical dome with radius 2.

Visualizing this:
- The boundary at the top is the sphere surface.
- The boundary at the bottom is the plane $z=0$.

Setting Up the Integral for Mass Calculation

1. Assumptions About Density

To compute the mass, we need to specify or assume the density $\rho(x,y,z)$. Typical choices include:
- Constant density: $\rho = \rho_0$,
- Radial density: $\rho(r) = \text{some function of } r$,
- Variable density depending on position.

For the sake of this analysis, we assume a constant density ρ_0, which simplifies calculations and is common in initial problems.

If the density is non-constant, the setup remains similar but with an integrand $\rho(x,y,z)$.

2. Volume Element in Cylindrical Coordinates

Given the symmetry of the problem (circular symmetry in the x, y plane), cylindrical coordinates are most convenient:

$$
\begin{cases}
x = r \cos \theta \\
y = r \sin \theta \\
z = z
\end{cases}
$$

with the Jacobian:

$$
dV = r \, dr \, d\theta \, dz
$$

The bounds:
- θ: $0 \leq \theta \leq 2\pi$,
- r: from 0 to the boundary at the sphere's surface,
- z: from 0 to the sphere surface.

Determining the Limits of Integration

1. Boundary in Terms of Coordinates

The sphere's surface:

$$
x^2 + y^2 + z^2 = 4 \implies r^2 + z^2 = 4
$$

For a fixed z, the maximum r is:

$$
r_{max}(z) = \sqrt{4 - z^2}
$$

Since z ranges from 0 to 2 (the hemisphere's top), the limits are:

$$
0 \leq z \leq 2
$$

and for each fixed z,

$$
0 \leq r \leq \sqrt{4 - z^2}
$$

Formulating the Volume Integral

The mass M is:

$$
M = \iiint_{V} \rho \, dV
$$

Assuming constant density ρ_0,

$$
M = \rho_0 \times \text{volume of the region}
$$

Expressed in cylindrical coordinates,

$$
M = \rho_0 \int_{\theta=0}^{2\pi} \int_{z=0}^{2} \int_{r=0}^{\sqrt{4 - z^2}} r \, dr \, dz \, d\theta
$$

Calculating the Integral Step-by-Step

1. Integrate with respect to r:

$$
\int_{0}^{\sqrt{4 - z^2}} r \, dr = \frac{1}{2} r^2 \bigg|_{0}^{\sqrt{4 - z^2}} = \frac{1}{2} (4 - z^2)
$$

2. Integrate with respect to z:

$$
\int_{0}^{2} \frac{1}{2} (4 - z^2) \, dz = \frac{1}{2} \int_{0}^{2} (4 - z^2) \, dz
$$

Calculating:

$$
\frac{1}{2} \left[4z - \frac{z^3}{3} \right]_0^2 = \frac{1}{2} \left(4 \times 2 - \frac{8}{3} \right) = \frac{1}{2} \left(8 - \frac{8}{3}\right) = \frac{1}{2} \times \left(\frac{24}{3} - \frac{8}{3}\right) = \frac{1}{2} \times \frac{16}{3} = \frac{8}{3}
$$

3. Integrate with respect to θ:

$$
\int_0^{2\pi} d\theta = 2\pi
$$

4. Final calculation of mass:

$$
M = \rho_0 \times 2\pi \times \frac{8}{3} = \frac{16\pi}{3} \rho_0
$$

Features, Pros, and Cons of the Approach

Features:
- The method leverages symmetry and cylindrical coordinates, simplifying the integral.
- Assumes constant density, which is common in basic physics and engineering problems.
- Provides a clear, step-by-step calculation process.

Pros:
- Straightforward setup makes it accessible for students.
- Can be extended easily to variable density functions.
- Highlights the importance of choosing appropriate coordinate systems.

Cons:
- Assumes uniform density; real-world applications may involve variable densities.
- The integral becomes more complex if the density depends on spatial variables.
- For more irregular shapes, this approach may not suffice, requiring numerical methods.

Extensions and Variations

1. Variable Density Functions

Suppose density varies with radius:

$$
\rho(r, \theta, z) = \rho_0 \times r^k
$$

then the integral becomes:

$$
M = \int_{0}^{2\pi} \int_{0}^{2} \int_{0}^{\sqrt{4 - z^2}} \rho_0 r^{k+1} dr dz d\theta
$$

which requires integrating power functions of r.

2. Different Boundary Conditions

If instead of a hemisphere, the region is a full sphere or a wedge, the limits change accordingly, and the integral setup must be adjusted.

3. Numerical Methods

For irregular shapes or complicated density functions, numerical integration (e.g., Monte Carlo methods, finite element analysis) can approximate the mass.

Conclusion

Calculating the mass of a crystal-shaped region in space, especially bounded above by a sphere and below by a plane, is a classical problem in multivariable calculus. By understanding the geometric boundaries, appropriately choosing coordinate systems, and carefully setting up the integrals, one can derive exact formulas for the mass under given density conditions.

In the specific case of a hemisphere of radius 2 with constant density, the mass simplifies to \(\frac{16\pi}{3} \rho_0\). This approach demonstrates the power of symmetry, coordinate transformations, and integral calculus in solving spatial volume problems.

Beyond this specific problem, the methodology applies broadly to diverse volume and mass calculations—serving

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