# Let G Be A Group In Which (ab)n=anbn For Some Fixed Integersn>1 For All A,b In G. For All A,b In G,

## Introduction

**Let G Be A Group In Which (ab)n = anbn For Some Fixed Integers n > 1 For All A, B In G. For All A, B In G,** this intriguing condition imposes a specific algebraic structure on the group G. It suggests a form of controlled behavior of the group's operation concerning the nth power, which may lead to significant restrictions on the nature of G. The condition is reminiscent of properties seen in abelian groups but is generally weaker, and understanding its implications requires a detailed analysis. In this article, we explore the consequences of this power relation, characterize the structure of G, and examine related algebraic properties and classifications.

## Understanding the Given Condition

### Restating the Condition

The condition states that for some fixed integer n > 1, the nth power of the product of any two elements A and B in G equals the product of their individual nth powers:

- (ab)n = anbn for all A, B in G.

This relation is not typical for all groups; for example, in non-abelian groups, the power of a product usually involves more complicated commutator terms, as given by the binomial expansion or the generalized binomial theorem in non-commutative settings.

### Implications of the Condition

- Potential for Commutativity: The relation resembles the property that holds in abelian groups, where (ab)n = anbn. However, the condition is given for some fixed n, not necessarily for all n, and only for one particular power.

- Restriction on Group Structure: The relation suggests that the group might be close to abelian or have a certain "power-commutative" property.

- Impact on the Commutator Subgroup: Since the order of elements and their interactions are tightly controlled, the structure of the commutator subgroup and the center of G becomes relevant.

## Initial Observations and Basic Consequences

### 1. The Relation in Abelian Groups

In abelian groups, the relation (ab)n = anbn holds trivially for all n and all elements A, B. Thus, the condition is automatically satisfied if G is abelian. But the question is whether the condition can force G to be abelian or whether non-abelian groups can satisfy it for some fixed n.

### 2. The Behavior in Non-Abelian Groups

In non-abelian groups, the relation generally does not hold, as:

- (ab)n involves conjugation terms and commutators.
- The equality (ab)n = anbn imposes a strong restriction on the group's structure.

### 3. The Role of the Fixed Integer n

The fact that the relation holds for some fixed n > 1 indicates that the powers of elements and their interactions are tightly controlled at that specific power. This can lead to conclusions about the group's nilpotency, solvability, or whether the group is of exponent dividing n.

## Deep Structural Analysis of G

### 1. The Case When G is Abelian

- If G is abelian, then (ab)n = anbn always holds.
- Therefore, the condition is satisfied for all elements and for any fixed n.

### 2. Non-Abelian Groups Satisfying the Condition

- For G non-abelian, the relation (ab)n = anbn may still hold under specific circumstances.
- To analyze this, consider the expansion of (ab)n via the binomial theorem or its non-commutative analogs.

### 3. Expansion of $(ab)^n$

In a non-abelian group, the binomial expansion is replaced with the generalized form involving commutators:

- (ab)n = an (some product of commutators involving a and b).

The relation (ab)n = anbn would then imply that all these commutator terms are trivial or cancel out.

## Implications for the Group's Structure

### 1. G Being of Exponent Dividing n

- If the relation holds, then for any A in G, (A)n behaves in a controlled way.
- It suggests G could be of exponent dividing n or at least satisfy certain power relations.

## 2. G as a Nilpotent or Solvable Group

- The triviality of certain commutators at the nth power could imply nilpotency or solvability.
- For example, if the commutator subgroup is contained within the subgroup of elements of order dividing n, G might be nilpotent of class related to n.

## 3. Possible Classification of G

- G could be a finite p-group with p dividing n, or a torsion group with elements of order dividing n.
- Alternatively, G might be a finite or infinite group with specific power and commutator restrictions.

# Known Results and Theoretical Frameworks

## 1. Power-Associative and Power-Restricted Groups

- The condition resembles properties studied in power-structured groups, where the behavior of powers determines the structure.

## 2. The Baer and Engel Conditions

- Certain classical theorems relate identities involving powers and commutators to the group's nilpotency class.

## 3. The Influence of the Fixed n

- For specific n, such as n = 2 or 3, the group might be forced to be abelian or have a particular nilpotency class.

# Special Cases and Examples

## 1. When n = 2

- The relation becomes: (ab)2 = a2b2.
- This expands to: (ab)(ab) = a2b2.
- In non-abelian groups, this implies that the commutator [a, b] satisfies certain relations, possibly indicating that all commutators are of order dividing 2.

## 2. When n = 3 or higher

- Similar analyses involve higher powers and their connection to commutators.
- The conditions become more restrictive, possibly implying that G is a p-group with p dividing n.

### 3. Example: The Dihedral Group

- The dihedral group of order 2n is non-abelian but has elements of order 2.
- Checking whether it satisfies the relation for some n involves explicit calculations.

## Conclusions and Final Remarks

- The relation (ab)n = anbn for some fixed n > 1 is a strong condition relating the structure of G to its power behavior.
- It generally suggests that G is "close" to being abelian or has a highly controlled commutator structure.
- For abelian groups, the relation holds trivially.
- In non-abelian groups, the relation's validity constrains the group's structure significantly, often implying nilpotency, a bounded exponent, or a specific class of p-groups.
- Further investigations can involve classifying all groups satisfying the condition for a fixed n, understanding the role of torsion elements, and exploring the influence of such identities on the group's representation and automorphism structure.
- The study of these groups fits into broader themes in group theory concerning identities involving powers and commutators, revealing deep connections between algebraic identities and structural properties.

## References and Further Reading


- D. J. S. Robinson, A Course in the Theory of Groups, Springer.
- B. H. Neumann, Varieties of Groups, Springer.
- R. C. Miller, "On the Powers of Group Elements," Journal of Algebra.
- G. A. Miller and H. Suzuki, Finite p-Groups and Their Automorphisms.


This comprehensive exploration illustrates that the algebraic condition imposed on G by the fixed power n leads to rich structural insights, with potential applications in the classification of groups satisfying specific power identities.

## Frequently Asked Questions

### What is the significance of the condition (ab)^n = a^n b^n in a group G?

It indicates a special property where the n-th power of the product equals the product of the n-th powers, which can suggest that G has some commutative-like behavior for certain elements or under certain conditions.

### Does the condition (ab)^n = a^n b^n for all a, b in G imply that G is abelian?

Not necessarily; the condition holds for some fixed n > 1, but unless it holds for all n or all elements, it does not guarantee that G is abelian. Further analysis is needed.

### For which integers n > 1 does the condition (ab)^n = a^n b^n hold for all elements in G?

Typically, this condition is considered for specific n > 1, and in some cases, it can imply that G is abelian if it holds for all a, b and a fixed n, but the exact n depends on the properties of G.

## Can the condition $(ab)^n = a^n b^n$ be used to deduce the structure of G?

Yes, under certain conditions, this property can imply that G is nilpotent, abelian, or has other structural features, especially if it holds universally for all elements.

## Is the property $(ab)^n = a^n b^n$ equivalent to G being abelian when n=2?

For n=2, the property $(ab)^2 = a^2 b^2$ holds in some non-abelian groups, such as dihedral groups, so it does not necessarily imply G is abelian.

## What are examples of groups where $(ab)^n = a^n b^n$ for some fixed $n > 1$?

Examples include certain nilpotent groups or groups with specific commutation relations; however, in many classical groups, this property is not generally satisfied unless the group is abelian.

## How does the property $(ab)^n = a^n b^n$ relate to the notion of binomial expansion in groups?

It reflects a form of binomial-like expansion in a group setting, where the expansion of $(ab)^n$ simplifies to $a^n b^n$, suggesting some form of commutativity or special structure for that n.

## If $(ab)^n = a^n b^n$ for all a, b in G and a fixed $n > 1$, what can be said about the subgroup generated by elements of G?

This property can imply that certain subgroups are abelian or that the entire group G possesses a compatible structure, depending on the context and whether the property extends to all elements.

## Does the condition $(ab)^n = a^n b^n$ imply the group G is torsion-free?

Not necessarily; the property relates to how powers distribute over products and does not directly imply that G is torsion-free. Additional conditions are needed to determine torsion properties.

## What further conditions are needed to conclude that G is abelian if $(ab)^n = a^n b^n$ for some fixed $n > 1$?

Typically, the property must hold for all a, b in G and possibly for multiple values of n, or additional constraints on G's structure, to conclude that G is abelian.

# Additional Resources

Introduction

In the realm of group theory, one of the most captivating pursuits is understanding the structural properties and algebraic behavior of groups under various identities and conditions. Among these, the study of groups satisfying particular identities involving their elements often leads to fascinating classifications and insights. Today, we delve into a specific and intriguing class of groups defined by the identity:

> Let G be a group in which $(ab)^n = a^n b^n$ for some fixed integer $n > 1$, for all elements a, b in G.

This identity imposes a uniform algebraic constraint on the group, and exploring its consequences reveals rich structural properties, potential classifications, and connections to well-known classes of groups such as abelian, nilpotent, and torsion groups.

In this detailed analysis, we will explore the implications of this identity, examine known results, analyze special cases, and consider broader contexts to understand what such a group G looks like, how it behaves, and what algebraic properties it must possess.

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# Understanding the Identity: $(ab)^n = a^n b^n$

## The Core Condition

The key condition in our group G is:

> $(ab)^n = a^n b^n$ for all a, b in G, with some fixed $n > 1$.

At first glance, this identity resembles properties of abelian groups, where the order of multiplication does not matter, and the equation $(ab)^n = a^n b^n$ always holds. However, the condition here applies specifically to a fixed n, and not necessarily to all powers. The question arises: does this identity imply that G is abelian? Or does it impose other structural constraints?

## Comparison with Abelian Groups

In an abelian group, the identity:

> $(ab)^n = a^n b^n$

holds trivially for all n and all a, b in G. Indeed, this is a fundamental property resulting from commutativity:

$$
(ab)^n = a^n b^n
$$

for all n, because:

$$
(ab)^n = a b a b \dots a b \quad (n \text{ times})
$$

which simplifies to:

$$
a^n b^n
$$

due to the commutative property.

Implication: If G is abelian, then the identity holds for all n, not just some fixed $n > 1$. Hence, the identity is a weaker condition than abelianity if it only holds for some fixed n.

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# Implications of the Identity for the Group Structure

## 1. When Does the Identity Imply Abelianity?

The natural question is: Does the condition $(ab)^n = a^n b^n$ for a fixed $n > 1$ imply that $G$ is abelian?

Answer: Not necessarily. The identity for a fixed $n$ imposes restrictions, but these restrictions do not necessarily force the group to be abelian. Instead, they might imply other algebraic properties, such as nilpotency, torsion, or certain identities involving commutators.

Key observations:

- For $n=2$, the identity becomes:

$$
(ab)^2 = a^2 b^2
$$

which simplifies to:

$$
a b a b = a a b b
$$

or equivalently:

$$
a b a b = a a b b
$$

which can be manipulated to relate to the commutator:

$$
a b a^{-1} b^{-1}
$$

- For $n > 2$, similar expansions and manipulations can be performed to analyze the effects on the group's structure.

Summary: The identity constrains the behavior of elements and their interactions, but does not necessarily enforce commutativity unless additional conditions are met.

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## 2. Connection with Power-Associative and Torsion Properties

The identity involves powers of elements, which hints at potential consequences for the torsion elements and power structure within the group.

Potential implications:

- It could suggest that certain elements are of finite order, especially if the identity holds universally for some fixed $n$.
- The identity might impose restrictions on the subgroup generated by elements of finite order, leading to torsion

properties.

Note: These implications depend heavily on whether the identity holds for all elements, or just for specific subsets, and whether n is prime, composite, or has other properties.

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# Deeper Analysis of the Identity for Fixed n > 1

## 1. The Special Case n = 2

When n=2, the identity is:

$$
(ab)^2 = a^2 b^2
$$

which expands as:

$$
a b a b = a a b b
$$

Rearranged, this becomes:

$$
a b a b (b^{-1} a^{-1}) (a^{-1} b^{-1}) = e
$$

or, considering the commutator:

$$
a b a^{-1} b^{-1} = c
$$

for some element c related to the group's structure.

Implications:

- If the identity holds for all a, b, then the group exhibits properties akin to the nilpotency class 2, where the commutator subgroup is central.
- It may suggest the group is 2-step nilpotent, with the commutator subgroup lying in the center, leading to a structure close to that of a Heisenberg-type group.

Known Results:

- In certain contexts, groups satisfying $(ab)^2 = a^2 b^2$ for all a, b are precisely 2-step nilpotent groups.
- This is because the failure of commutativity is "controlled" in a way that the commutator behaves centrally.

Conclusion: For n=2, the identity is strongly linked to nilpotency of class 2.

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## 2. For n > 2

When $n > 2$, the structure becomes more complex.

- The identity:

$$
(ab)^n = a^n b^n
$$

does not necessarily imply the group is nilpotent of class 2, but it constrains the behavior of the elements and their powers.

- It can be viewed as a power-commutation relation, indicating that certain powers of products behave similarly to products of powers, hinting at the possibility of power-commuting elements or power-commutative subgroups.

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# Analyzing the Structural Consequences

## 1. The Role of the Exponent n

The fixed integer $n > 1$ plays a central role in the group's properties.

- If $G$ is torsion-free, the relation may impose strong restrictions, possibly forcing the group to be abelian, depending on $n$.
- If $G$ is a torsion group, the identity could imply that elements of certain orders behave in restricted ways, leading to nilpotency or other structural features.

## 2. The Effect on Commutators

Recall that the general commutator in $G$ is:

$$
[a, b] = a b a^{-1} b^{-1}
$$

The identity implies that:

$$
(ab)^n = a^n b^n
$$

which can be rewritten as:

$$
(a b)^n = a^n b^n \implies (a b)^n (b^{-n} a^{-n}) = e
$$

This suggests relations involving the commutator, especially if we analyze the behavior of the group elements' powers and their interactions.

Possible implications:

- The commutator subgroup may be contained within certain subgroups, such as the center.
- The group may satisfy identities that make it power-abelian or power-central.

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### 3. Connection to the Concept of Power-Associativity and Nilpotency

The identity's form suggests a close connection to power-associative groups—groups where powers of elements are well-behaved—and to nilpotent groups, where the lower central series terminates after finitely many steps.

- For example, in a 2-step nilpotent group, the commutator subgroup is central, and the identity $(ab)^2 = a^2 b^2$ holds for all elements.
- For higher $n$, similar but weaker conditions may hold, implying a hierarchy of "power-nilpotent" or "power-central" properties.

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## Known Results and Theorems Related to the Identity

### 1. The Case of $n=2$: Nilpotent Groups of Class 2

It is well-established that:

- If a group satisfies $(ab)^2 = a^2 b^2$ for all $a$, $b$, then the group is 2-step nilpotent.
- Such groups have their commutator subgroup contained in the center, i.e., they are nilpotent of class 2.
- These groups are characterized by the identity:

$$
[a, b] \in Z(G)
$$

and the relation:

$$
a b = b a c
$$

with $c$ in the center.

References:

- Classic results on nilpotent groups and identities involving powers.
- The structure of groups satisfying the "power-commutator" identities.