Let R = Z[-5). Decide Whether Or Not 11 Is An Irreducible Element Of R And Whether Or Not < 11 >

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Understanding the properties of elements within algebraic structures like rings is fundamental in abstract algebra. In particular, analyzing whether a given element is irreducible or whether a principal ideal generated by that element is prime or not offers insight into the structure and factorization properties of the ring. Here, we explore the ring \(R = \mathbb{Z}[\sqrt{-5}] \), often called the ring of integers in the quadratic field \(\mathbb{Q}(\sqrt{-5}) \), and examine whether the element 11 is irreducible within this ring. Additionally, we analyze the nature of the ideal \(\langle 11 \rangle \) generated by 11, determining if it is prime or not.

Background on the Ring \(\mathbb{Z}[\sqrt{-5}] \)

Definition and Basic Properties

The ring \(R = \mathbb{Z}[\sqrt{-5}] \) consists of all numbers of the form:

\[
a + b\sqrt{-5} \quad \text{with } a, b \in \mathbb{Z}
\]

This ring is an integral domain, meaning it has no zero divisors, but unlike the ring of integers \(\mathbb{Z} \), it is not a Unique Factorization Domain (UFD). The failure of unique factorization in \(\mathbb{Z}[\sqrt{-5}] \) is a famous example often discussed in algebraic number theory.

Norm Function

A key concept for analyzing elements in quadratic rings is the norm:

\[
N(a + b\sqrt{-5}) = a^2 + 5b^2
\]

This norm is multiplicative:

\[
N(\alpha \beta) = N(\alpha) N(\beta)
\]

and helps determine the reducibility or primality of elements.

Is 11 Irreducible in $\mathbb{Z}[\sqrt{-5}]$?

Understanding Irreducibility

An element $\alpha \in R$ is irreducible if it is not a unit (invertible) and cannot be factored into non-unit elements:

$$
\alpha = \beta \gamma
$$

where both β, γ are non-units in R.

Since 11 is a rational prime, the question reduces to whether it factors into non-unit elements within R.

Norm of 11 and its Implications

Calculate the norm:

$$
N(11) = 11^2 = 121
$$

Any factorization of 11 in R:

$$
11 = \alpha \beta
$$

must satisfy:

$$
N(11) = N(\alpha) N(\beta) = 121
$$

Because $N(\alpha), N(\beta) \in \mathbb{Z}$, their positive integer factors are divisors of 121.

The divisors of 121 are:

$$
1, 11, 121
$$

- If $N(\alpha) = 1$, then α is a unit (since the only elements with norm 1 are units).

- If $N(\alpha) = 11$, then $N(\beta) = 11$.

- If $N(\alpha) = 121$, then $N(\beta) = 1$.

This suggests that if 11 factors non-trivially, the potential factors must have norms 1 or 11.

Are There Elements with Norm 11 in R?

We check whether there exists $\alpha = a + b \sqrt{-5}$ such that:

$$
N(\alpha) = a^2 + 5b^2 = 11
$$

Solve for integer solutions:

$$
a^2 + 5b^2 = 11
$$

Possible integer pairs:

- $b = 0$:

$$
a^2 = 11 \Rightarrow a^2 = 11
$$

No integer solution.

- $b = \pm 1$:

$$
a^2 + 5 = 11 \Rightarrow a^2 = 6
$$

No integer solution.

- $b = \pm 2$:

$$
a^2 + 20 = 11 \Rightarrow a^2 = -9
$$

Impossible.

- Larger $|b|$ will only increase $5b^2$, exceeding 11.

Thus, no elements in R have norm 11.

Conclusion on Irreducibility of 11

Since no element in R has norm 11, and 11 is a rational prime, the only possibility for 11 to factor in R would be if it could be written as a product of elements with norms dividing 121, specifically with norm 1 or 11. But because no element has norm 11, and 11 itself has norm 121, the only factors are units and associates.

- Units in R: Elements with norm 1. For $\mathbb{Z}[\sqrt{-5}]$, the only units are ± 1 because:

$$
N(\pm 1) = 1
$$

- Associates: Elements differing by multiplication by a unit.

Therefore, 11 cannot be factored into non-unit elements of R, implying that 11 is irreducible in R.

Analyzing the Ideal $\langle 11 \rangle$

Prime vs. Maximal Ideals

In algebraic number rings, the nature of the ideal $\langle p \rangle$, where p is a prime in $\mathbb{Z}$, is important. An ideal is:

- Prime if the quotient ring $R / \langle p \rangle$ is an integral domain.
- Maximal if the quotient is a field.

In the case of $R = \mathbb{Z}[\sqrt{-5}]$, the ideal $\langle 11 \rangle$ is prime if and only if 11 remains prime in R.

Factorization of 11 in R

Since 11 is irreducible in R, and R is a Dedekind domain (an integral domain where every non-zero prime ideal factors uniquely into prime ideals), the ideal $\langle 11 \rangle$ is either:

- a prime ideal itself, or
- factors into a product of prime ideals.

Given that 11 is irreducible but not necessarily prime, we analyze whether $\langle 11 \rangle$ is prime.

Prime Decomposition of $\langle 11 \rangle$

In quadratic integer rings, the factorization of primes depends on how the prime splits in the field extension. The splitting behavior is determined by the quadratic residue symbol.

- The ideal $\langle p \rangle$ in R factors depending on whether

p is:

- Inert: remains prime.

- Splits: factors into two distinct prime ideals.

- Ramifies: factors as the square of a prime ideal.

Use quadratic reciprocity to analyze:

$$
\left(\frac{-5}{p} \right)
$$

which indicates whether p splits in $\mathbb{Q}(\sqrt{-5})$.

For $p = 11$:

$$
\left(\frac{-5}{11} \right)
$$

The Legendre symbol:

$$
\left(\frac{-5}{11} \right) = \left(\frac{-1}{11} \right) \times \left(\frac{5}{11} \right)
$$

- $\left(\frac{-1}{11} \right) = (-1)^{(11-1)/2} = (-1)^5 = -1$

- $\left(\frac{5}{11} \right)$:

Since 11 is prime, use quadratic reciprocity:

$$
\left(\frac{5}{11} \right) = \left(\frac{11}{5} \right) \times (-1)^{\frac{(5-1)(11-1)}{4}}
$$

Calculate:

$$
\left(\frac{11}{5} \right) = \left(\frac{1}{5} \right) = 1
$$

because $11 \equiv 1 \pmod{5}$.

Exponents:

$$
\frac{(5-1)(11-1)}{4} = \frac{4 \times 10}{4} = 10
$$

and

$$
(-1)^{10}

Frequently Asked Questions

What is the ring R = Z[−5), and how is it defined?

The ring R = Z[−5) typically refers to the set of integers greater than or equal to −5, i.e., R = {−5, −4, −3, −2, −1, 0, 1, 2, 3, 4, 5, ...}, forming a subset of the integers including all elements from −5 upwards.

Is 11 an element of the ring R = Z[−5)?

Yes, 11 is an element of R since R includes all integers greater than or equal to −5, and 11 is greater than −5.

What does it mean for 11 to be an irreducible element in R?

In the context of ring R, 11 is irreducible if it is a non-unit element that cannot be factored into two non-unit elements within R, meaning it cannot be expressed as a product of two elements neither of which is a unit.

Is 11 a prime element in the ring R = Z[−5)?

In the ring of integers, 11 is prime, and since R is a subring of the integers, 11 remains prime in R, provided that its divisibility properties are preserved within R.

Does the ideal < 11 > in R consist of all multiples of 11 within R?

Yes, the ideal < 11 > in R includes all elements of R that are multiples of 11, i.e., elements of the form 11k where k is in R.

Is the element 11 a unit in R?

No, 11 is not a unit in R because its multiplicative inverse does not exist within R, as R consists of integers greater than or equal to −5, and 11's inverse is not an integer.

How can we determine if 11 is irreducible in R?

Since 11 is a prime in Z and R contains 11, and if R is an integral domain similar to Z, then 11 is irreducible in R as it cannot be factored into non-unit elements within R.

Are all prime elements in R also irreducible?

Yes, in integral domains like R, all prime elements are necessarily irreducible, but the converse may not always hold unless R has specific properties.

Does the ideal < 11 > equal the set of all multiples of 11 in R?

Yes, the ideal < 11 > in R consists exactly of all elements in R that are multiples of 11, i.e., elements of the form 11k, where k is in R.

What implications does the irreducibility of 11 have for the structure of the ideal < 11 >?

If 11 is irreducible and prime in R, then the ideal < 11 > is a prime ideal, which influences the factorization properties within R and the structure of its quotient rings.

Additional Resources

Algebraic Structures: Analyzing Irreducibility of 11 in R = Z[-5]

Introduction to the Ring R = Z[−5]

Imagine a universe where integers intertwine with algebraic numbers, creating a rich tapestry of mathematical relationships. This universe is encapsulated beautifully within the ring R = Z[-5], a subset of the quadratic field $Q(\sqrt{-5})$.

This ring comprises all numbers of the form $a + b\sqrt{-5}$, where a and b are integers. Formally, it is expressed as:

\[
R = \{ a + b\sqrt{-5} \mid a, b \in \mathbb{Z} \}
\]

Understanding the properties of elements within R, particularly whether certain elements are irreducible or prime, is fundamental in algebraic number theory. This exploration aims to analyze whether 11 is an irreducible element in R and to examine the structure of the ideal <11>.

Understanding the Underlying Concepts

Before delving into specifics, it is crucial to clarify key algebraic concepts: irreducibility, primality, and ideals in the context of rings like R.

Irreducible Elements

An element α in a ring R is called irreducible if:

- It is not a unit (i.e., it does not have a multiplicative inverse in R),
- Whenever $\alpha = \beta\gamma$ for some $\beta, \gamma \in R$, then at least one of β or γ is a unit.

In essence, an irreducible element cannot be factored into non-unit elements, indicating its fundamental 'building block' status.

Prime Elements and Their Connection to Irreducibility

A prime element p in R satisfies:

- p is not a unit,
- If $p \mid xy$ (meaning p divides the product xy), then $p \mid x$ or $p \mid y$.

In many rings, especially Unique Factorization Domains (UFDs), irreducibility and primality coincide. However, in rings like $Z[\sqrt{-5}]$, which are not UFDs, these concepts diverge, making the analysis more intriguing.

Ideals and Their Significance

An ideal in R generated by an element a is the set:

$$
< a > = \{ r \cdot a \mid r \in R \}
$$

The structure of these ideals, especially whether they are prime or maximal, reflects the divisibility properties within R.

The Nature of R = Z[−5]

Quadratic integer rings such as $Z[\sqrt{-5}]$ display complex behavior. Unlike the familiar integers Z, which form a UFD, $Z[\sqrt{-5}]$ is known to be not a UFD. This impacts the factorization of elements like 11 and the nature of the ideals they generate.

Key facts about $Z[\sqrt{-5}]$:

- It is an order in the quadratic field $Q(\sqrt{-5})$.
- It has class number greater than 1, indicating the existence of non-principal ideals.
- Some elements, like 6, factor uniquely, whereas others, like 11, may not.

Analyzing the Element 11 in R = $Z[\sqrt{-5}]$

The core question is: Is 11 an irreducible element in R? To answer this, we follow a systematic approach involving norm calculations and factorization attempts.

Calculating the Norm of 11

The norm function $N: R \to Z$ is critical in algebraic number theory, defined as:

$$
N(a + b\sqrt{-5}) = a^2 + 5b^2
$$

Since 11 is an integer, its norm in R is simply:

$$
N(11) = 11^2 = 121
$$

This provides a lower bound for possible factorizations, as any factorization of 11 into non-unit elements must respect the multiplicative property of norms:

$$
N(\alpha\beta) = N(\alpha) \cdot N(\beta)
$$

Attempting to Factor 11 in R

To determine whether 11 is reducible, we need to see if it can be expressed as a product of two non-unit elements in R, say:

$$
11 = \alpha \cdot \beta
$$

with $\alpha, \beta \in R$ and neither being units.

Given the norm's multiplicative property:

$$
N(\alpha) \cdot N(\beta) = 121
$$

Since $N(\alpha)$ and $N(\beta)$ are positive integers, the possible pairs are the divisors of 121:

- 1 and 121
- 11 and 11

Units in R are elements with norm 1. For $Z[\sqrt{-5}]$, units are only ±1, because:

$$
N(\pm1) = 1
$$

Thus, the possible norms for non-unit factors are 11 or 121.

Checking for Elements of Norm 11

To factor 11, we look for elements in R with norm 11, i.e., solutions to:

$$
a^2 + 5b^2 = 11
$$

where a, b ∈ Z.

Let's analyze this Diophantine equation:

- For b=0:

$$
a^2 = 11 \Rightarrow a^2=11
$$

which has no integer solutions.

- For b=±1:

$$
a^2 + 5(1)^2 = a^2 + 5 = 11 \Rightarrow a^2=6
$$

No integer solutions.

- For b=±2:

$$
a^2 + 5(4) = a^2 + 20 = 11 \Rightarrow a^2=-9
$$

Impossible.

- For b=±3:

$$
a^2 + 45=11 \Rightarrow a^2=-34
$$

Impossible.

Since larger |b| will only increase the left side, no solutions exist with b ≠ 0.

Conclusion: There are no elements in R with norm 11.

Implication for Factorization

Since no element in R has norm 11, the only way 11 could factor in R is if it factors into elements with norms dividing 121 (i.e., 1, 11, or 121):

- Units (norm 1): ±1, which are trivial units.
- Elements with norm 11: none exist.
- Elements with norm 121: potential candidates.

Now, let's analyze whether 11 divides an element with norm 121 and whether that element can be factored further.

Factorization of 11 in R

Key observation: Since there are no elements with norm 11, 11 cannot be written as a product of two non-unit elements with smaller norms. It can only be associated with elements of norm 121, which suggests that:

$$
11 \sim \pm \text{(element with norm 121)}
$$

But what are the elements with norm 121?

Solve:

$$
a^2 + 5b^2 = 121
$$

for integers a, b.

- For b=0:

$$
a^2=121 \Rightarrow a= \pm 11
$$

- For b=±1:

$$
a^2 + 5=121 \Rightarrow a^2=116
$$

No integer solution.

- For b=±2:

$$
a^2 + 20=121 \Rightarrow a^2=101
$$

No.

- For $b=\pm 3$:

$$
a^2 + 45=121 \Rightarrow a^2=76
$$

No.

- For $b=\pm 4$:

$$
a^2 + 80=121 \Rightarrow a^2=41
$$

No.

- For $b=\pm 5$:

$$
a^2 + 125=121 \Rightarrow a^2=-4
$$

No.

Higher $|b|$ values only increase the left side, so no solutions.

Thus, the only elements with norm 121 are:

$$
a + b \sqrt{-5} \quad \text{with} \quad a=\pm 11, \quad b=0
$$

which are just ± 11 in the integers.

Conclusion on Factorization

Since 11 is associated with the element 11 itself, and there are no elements with norm 11, it follows that:

- 11 cannot be factored into non-unit elements in R.
- The only divisors are units ± 1 and ± 11.

Thus, 11 is an irreducible element in R.

Is 11 a Prime Element in R?

While we've established that

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