Problem 2. (1 Point)Consider The Initial Value Problemy" + 4y = 16t,y(0) 9, Y(0) 6.a. Take The Laplace

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Introduction

Solving initial value problems (IVPs) involving differential equations is a fundamental aspect of applied mathematics and engineering. One effective method for tackling such problems, especially linear ordinary differential equations with constant coefficients, is the Laplace transform. In this article, we will explore Problem 2, which involves solving the differential equation:

$$ y'' + 4y = 16t, \quad y(0) = 9, \quad y'(0) = 6 $$

using the Laplace transform technique. This approach converts the differential equation into an algebraic equation in the Laplace domain, simplifying the process of finding the solution that satisfies the initial conditions.

Understanding the Initial Value Problem

The Differential Equation

The differential equation:

$$ y'' + 4y = 16t $$

is a nonhomogeneous second-order linear differential equation. It consists of:

- A homogeneous part: $ y'' + 4y = 0 $
- A nonhomogeneous term: $ 16t $

The presence of initial conditions $ y(0) = 9 $ and $ y'(0) = 6 $ enables us to determine the particular solution that satisfies the specific problem setup.

Significance of Initial Conditions

Initial conditions are essential for uniquely determining the solution of a differential equation. They specify the state of the system at a particular point—in this case, at $t=0$:

- $y(0) = 9$: the initial position or value of the function
- $y'(0) = 6$: the initial velocity or rate of change of the function

These conditions are incorporated into the Laplace transform solution process to ensure the final answer is tailored to the problem.

Applying the Laplace Transform

What is the Laplace Transform?

The Laplace transform converts a time-domain function $y(t)$ into a complex frequency domain function $Y(s)$:

$$\mathcal{L}\{ y(t) \} = Y(s) = \int_0^{\infty} e^{-st} y(t) \, dt$$

This transformation simplifies differential equations into algebraic equations because derivatives in the time domain correspond to polynomial expressions in the s-domain.

Transforming the Differential Equation

Applying the Laplace transform to the entire differential equation:

$$y'' + 4y = 16t$$

we utilize the properties:

- $\mathcal{L}\{ y'' \} = s^2 Y(s) - s y(0) - y'(0)$
- $\mathcal{L}\{ y \} = Y(s)$
- $\mathcal{L}\{ t \} = \frac{1}{s^2}$

Transforming each term:

$$s^2 Y(s) - s y(0) - y'(0) + 4Y(s) = 16 \cdot \frac{1}{s^2}$$

Substituting the initial conditions $y(0)=9$, $y'(0)=6$:

$$s^2 Y(s) - 9s - 6 + 4Y(s) = \frac{16}{s^2}$$

Solving for $Y(s)$

Rearranging the Equation

Group the terms involving $Y(s)$:

$$ (s^2 + 4) Y(s) = 9s + 6 + \frac{16}{s^2} $$

Expressed explicitly:

$$ Y(s) = \frac{9s + 6 + \frac{16}{s^2}}{s^2 + 4} $$

To facilitate inverse Laplace transformation, rewrite $Y(s)$ as a single rational expression:

$$ Y(s) = \frac{(9s + 6)s^2 + 16}{s^2 (s^2 + 4)} $$

which simplifies to:

$$ Y(s) = \frac{9s^3 + 6s^2 + 16}{s^2 (s^2 + 4)} $$

Partial Fraction Decomposition

Why Partial Fractions?

Partial fraction decomposition allows us to break down complex rational functions into simpler terms whose inverse Laplace transforms are well-known.

Decomposition Process

Express $Y(s)$ as:

$$ \frac{9s^3 + 6s^2 + 16}{s^2 (s^2 + 4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + 4} $$

Multiply through by the denominator to eliminate fractions:

$$ 9s^3 + 6s^2 + 16 = A s (s^2 + 4) + B (s^2 + 4) + (C s + D) s^2 $$

Simplify the right side:

$$ 9s^3 + 6s^2 + 16 = A s^3 + 4A s + B s^2 + 4 B + C s^3 + D s^2 $$

Group similar powers of s:

$$ (A + C) s^3 + (B + D) s^2 + 4A s + 4 B $$

Matching coefficients from both sides:

- Coefficient of s^3:

$$ A + C = 9 $$

- Coefficient of s^2:

$$ B + D = 6 $$

- Coefficient of s:

$$ 4A = 0 \Rightarrow A=0 $$

- Constant term:

$$ 4 B = 16 \Rightarrow B=4 $$

Using $A=0$, find C:

$$ 0 + C = 9 \Rightarrow C=9 $$

Using $B=4$, find D:

$$ 4 + D = 6 \Rightarrow D=2 $$

Inverse Laplace Transform

Final Expression for $Y(s)$

Now, rewrite $Y(s)$:

$$ Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C s + D}{s^2 + 4} = \frac{0}{s} + \frac{4}{s^2} + \frac{9 s + 2}{s^2 + 4} $$

which simplifies to:

$$ Y(s) = \frac{4}{s^2} + \frac{9 s + 2}{s^2 + 4} $$

Applying Inverse Laplace

Using standard inverse transforms:

- $\mathcal{L}^{-1}\left\{ \frac{1}{s^2} \right\} = t$
- $\mathcal{L}^{-1}\left\{ \frac{s}{s^2 + a^2} \right\} = \cos a t$
- $\mathcal{L}^{-1}\left\{ \frac{1}{s^2 + a^2} \right\} = \frac{\sin a t}{a}$

Express the second term:

$$ \frac{9 s + 2}{s^2 + 4} = 9 \cdot \frac{s}{s^2 + 4} + 2 \cdot \frac{1}{s^2 + 4} $$

Inverse transforms:

- $\mathcal{L}^{-1}\left\{ \frac{s}{s^2 + 4} \right\} = \cos 2 t$
- $\mathcal{L}^{-1}\left\{ \frac{1}{s^2 + 4} \right\} = \frac{\sin 2 t}{2}$

\)

Thus, the solution $y(t)$:

$$
\begin{aligned}
y(t) &= 4 t + 9 \cos 2 t + 2 \cdot \frac{\sin 2 t}{2} \\
&= 4 t + 9 \cos 2 t + \sin 2 t
\end{aligned}
$$

Final Solution and Interpretation

The solution to the initial value problem:

$$ y(t) = 4 t + 9 \cos 2 t + \sin 2 t $$

satisfies the differential equation $y'' + 4 y = 16 t$ with initial conditions $y(0) = 9$ and $y'(0) = 6$.

Significance of the Laplace Transform Method

Using the Laplace transform simplifies the process of solving linear differential equations with initial conditions. It transforms the problem into algebraic manipulation, which is often more straightforward than solving the differential equation directly, especially when dealing with nonhomogeneous terms.

Summary

In this article, we:

- Introduced the initial value problem involving a second-order linear differential equation.
- Applied the Laplace transform to convert the differential equation into an algebraic equation.
- Solved for the Laplace domain function $Y(s)$ using

Frequently Asked Questions

What is the main goal when solving the initial value problem 4y' + y = 16t with y(0) = 6 and y'(0) = 9

using Laplace transforms?

The main goal is to find the explicit solution y(t) that satisfies the differential equation along with the given initial conditions by applying Laplace transforms to convert the problem into an algebraic equation.

How do you apply the Laplace transform to the differential equation 4y' + y = 16t?

Apply the Laplace transform to both sides, using linearity: L{4y'} + L{y} = L{16t}. Recall that L{y'} = sY(s) - y(0), where Y(s) is the Laplace transform of y(t).

What initial conditions are used in the Laplace transform method for this problem?

The initial conditions y(0) = 6 and y'(0) = 9 are used to evaluate expressions involving the Laplace transforms, specifically to handle derivatives in the transformed domain.

How do you handle the term involving y' in the Laplace domain for this problem?

Use the property L{y'} = sY(s) - y(0). Since y(0) = 6, replace accordingly to incorporate the initial condition into the algebraic equation.

What is the next step after taking the Laplace transform of the differential equation?

Solve for Y(s) in the algebraic equation obtained, then take the inverse Laplace transform to find y(t).

What challenges might arise when taking the inverse Laplace transform in this problem?

Potential challenges include managing complex expressions for Y(s), partial fraction decomposition, and ensuring the inverse transform corresponds to a physically meaningful solution that satisfies the initial conditions.

Additional Resources

Laplace Transform: Unlocking the Power of Differential Equations

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Introduction: The Significance of the Laplace Transform in Differential

Equations

In the realm of mathematical analysis, differential equations serve as fundamental tools for modeling real-world phenomena—ranging from physics and engineering to economics and biological systems. Among various techniques for solving these equations, the Laplace transform has established itself as an exceptionally powerful method, especially for initial value problems involving linear ordinary differential equations (ODEs). Its ability to convert complex differential equations into algebraic equations simplifies the problem significantly, thereby making seemingly intractable problems manageable.

This article aims to provide an in-depth, comprehensive exploration of the application of the Laplace transform to a specific initial value problem:

$$
\text{Consider } y'' + 4y = 16t, \quad y(0) = 9, \quad y'(0) = 6.
$$

We will examine how to effectively utilize the Laplace transform to solve this differential equation, step by step, elucidating each process with clarity and precision. Whether you're a student striving to master the technique or an enthusiast exploring the depths of differential equations, this detailed review offers valuable insights into the power and elegance of the Laplace transform.

Understanding the Initial Value Problem

The Differential Equation

The problem involves a second-order linear differential equation:

$$
y'' + 4y = 16t,
$$

where $y = y(t)$ is the unknown function of time t. The right-hand side $16t$ introduces a nonhomogeneous component, making the solution more interesting and challenging.

Initial Conditions

Given initial conditions are:

$$
y(0) = 9, \quad y'(0) = 6.
$$

These initial conditions are crucial, as they allow us to solve for any arbitrary constants that appear in the general solution.

The Rationale for Using the Laplace Transform

The primary motivation for employing the Laplace transform is to simplify the process of solving differential equations, especially those with initial conditions and nonhomogeneous terms. The transform converts derivatives in the time domain into algebraic terms in the complex frequency domain, which can be manipulated more straightforwardly.

Key advantages include:

- Handling initial conditions naturally: The Laplace transform inherently incorporates initial conditions into the transformed algebraic equations.
- Simplification of nonhomogeneous terms: The transform of functions like t, e^{at}, or sinusoidal functions is well-understood.
- Streamlining the solution process: Instead of solving the differential equation directly, we solve an algebraic equation and then invert to find $y(t)$.

Step-by-Step Solution Using the Laplace Transform

Step 1: Applying the Laplace Transform to Both Sides

Recall the Laplace transform notation:

$$
\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}f(t)\,dt.
$$

Applying the transform to each term:

$$
\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = 16\mathcal{L}\{t\}.
$$

Using known properties:

$$
\mathcal{L}\{y''\} = s^2 Y(s) - s y(0) - y'(0),
$$
$$
\mathcal{L}\{y\} = Y(s),
$$
$$
\mathcal{L}\{t\} = \frac{1}{s^2}.
$$

Substituting initial conditions:

$$
s^2 Y(s) - s \times 9 - 6 + 4 Y(s) = 16 \times \frac{1}{s^2}.
$$

This simplifies the differential equation into an algebraic form.

Step 2: Formulating the Algebraic Equation

Group the $Y(s)$ terms:

$$
(s^2 + 4) Y(s) = s \times 9 + 6 + \frac{16}{s^2}.
$$

Expressed explicitly:

$$
(s^2 + 4) Y(s) = 9s + 6 + \frac{16}{s^2}.
$$

To isolate $Y(s)$:

$$
Y(s) = \frac{9s + 6 + \frac{16}{s^2}}{s^2 + 4}.
$$

The goal now is to perform the inverse Laplace transform to find $y(t)$.

Simplifying the Expression for $Y(s)$

Step 3: Rationalizing the Expression

To facilitate inverse transformation, rewrite the numerator to combine terms over a common denominator:

$$
Y(s) = \frac{(9s + 6) s^2 + 16}{s^2 (s^2 + 4)}.
$$

Expanding numerator:

$$
(9s + 6) s^2 + 16 = 9s^3 + 6 s^2 + 16.
$$

Therefore:

$$
Y(s) = \frac{9s^3 + 6 s^2 + 16}{s^2 (s^2 + 4)}.
$$

Step 4: Partial Fraction Decomposition

The expression is rational, so partial fractions are appropriate for inverse transformation. Set:

$$
Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + 4}.
$$

Multiply both sides by $s^2 (s^2 + 4)$:

$$
9s^3 + 6 s^2 + 16 = A s (s^2 + 4) + B (s^2 + 4) + (C s + D) s^2.
$$

Expanding:

$$
9s^3 + 6 s^2 + 16 = A s^3 + 4 A s + B s^2 + 4 B + C s^3 + D s^2.
$$

Group like terms:

$$
\text{Coefficients of } s^3: \quad A + C,
$$
$$
\text{Coefficients of } s^2: \quad B + D,
$$
$$
\text{Coefficients of } s: \quad 4A,
$$
$$
\text{Constant term: } 4B.
$$

Matching coefficients with the left side:

$$
\begin{cases}
A + C = 9, \\
B + D = 6, \\
4A = 0 \Rightarrow A=0, \\
4 B = 16 \Rightarrow B=4.
\end{cases}
$$

From $4A=0$, $A=0$. Then:

$$
A + C = 9 \Rightarrow 0 + C=9 \Rightarrow C=9,
$$
$$
B + D=6 \Rightarrow 4 + D=6 \Rightarrow D=2.
$$

Step 5: Final Partial Fraction Decomposition

Putting all together:

$$
Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C s + D}{s^2 + 4} = \frac{0}{s} + \frac{4}{s^2} + \frac{9 s + 2}{s^2 + 4}.
$$

Simplify:

$$
Y(s) = \frac{4}{s^2} + \frac{9 s + 2}{s^2 + 4}.
$$

Inverse Laplace Transform to Find $y(t)$

Step 6: Recognizing Standard Inverse Transforms

Recall the standard inverse Laplace transforms:

- $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$,
- $\mathcal{L}^{-1}\left\{\frac{s}{s^2 + a^2}\right\} = \cos a t$,
- $\mathcal{L}^{-1}\left\{\frac{1}{s^2 + a^2}\right\} = \frac{\sin a t}{a}$.

Apply these to each term:

- $\mathcal{L}^{-1}\left\{\frac{4}{s^2}\right\} = 4 t$,
- $\mathcal{L}^{-1}\left\{\frac{9 s}{s^2 + 4}\right\} = 9 \cos 2 t$,
- $\mathcal{L}^{-1}\left\{\frac{2}{s^2 + 4}\right\} = 2 \times \frac{\sin 2 t}{2} = \sin 2 t$.

Step 7: Final Solution

Combining all parts:

$$
y(t) = 4 t + 9 \cos 2 t + \sin 2 t + C,
$$

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where \( C \) is the constant of integration, which is determined by initial
conditions. Since the inverse Laplace transform inherently satisfies initial
conditions, the solution derived from the transform already accounts for \(
y(0) = 9 \) and \( y'(0) =
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