Problem 2 [6 Marks; 3 Each] 2.1 Express The Surface Area Of The Portion Of The Paraboloid 2z = X2 + Y2

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Understanding how to compute the surface area of a paraboloid is fundamental in multivariable calculus, especially when dealing with surfaces defined implicitly or parametrically. This article provides a comprehensive guide to expressing the surface area of the particular paraboloid $2z = x^2 + y^2$, focusing on the mathematical derivation, the parametric representation, and the integral setup needed to evaluate such an area. Whether you're a student preparing for exams or a professional working on related applications, mastering this process is essential for tackling surface area problems efficiently.

Introduction to Surface Area of Surfaces in Multivariable Calculus

Surface area calculations are a key aspect of multivariable calculus, with applications in physics, engineering, and computer graphics. When dealing with surfaces like paraboloids, spheres, cylinders, or complex geometric shapes, expressing the surface area involves understanding the surface's parametrization and applying the surface integral formula.

Key concepts include:
- Parametric representation of surfaces
- Surface element dS
- Surface area integral

Understanding these concepts allows for a systematic approach to calculating the surface area of various surfaces, including paraboloids, which are common in real-world modeling.

Understanding the Paraboloid $2z = x^2 + y^2$

The paraboloid $2z = x^2 + y^2$ is a classic example of a quadratic surface. It opens upward with its vertex at the origin and is symmetric about the z-axis.

Characteristics of this paraboloid:

- Symmetric about the z-axis
- Opens upward from the vertex at (0,0,0)
- The surface can be described in a more manageable form for calculations

Expressed explicitly, the surface can be written as:
$$ z = \frac{x^2 + y^2}{2} $$

This explicit form simplifies the setup of the surface integral, especially when switching to polar coordinates.

Parametric Representation of the Surface

To compute the surface area, the first step is to find a suitable parametric form.

Using polar coordinates:
- Let $x = r \cos \theta$
- Let $y = r \sin \theta$

Substituting into the surface equation:
$$ z = \frac{r^2}{2} $$

Thus, the parametric form of the surface S can be written as:
$$
\mathbf{r}(r, \theta) = \langle r \cos \theta, r \sin \theta, \frac{r^2}{2} \rangle
$$
where:
- $r \geq 0$
- $0 \leq \theta \leq 2\pi$

Note: The limits for r depend on the specific portion of the paraboloid under consideration, often dictated by a boundary curve or a maximum radius.

Deriving the Surface Area Formula

The surface area S of a parametric surface $\mathbf{r}(u, v)$ is given by the surface integral:
$$
S = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv
$$
where:
- D is the domain in the uv-plane
- $\mathbf{r}_u$, $\mathbf{r}_v$ are partial derivatives of $\mathbf{r}$ with

respect to u and v

Applying this to our surface:
- $u = r$
- $v = \theta$

Calculating the derivatives:

$$
\mathbf{r}_r = \frac{\partial \mathbf{r}}{\partial r} = \langle \cos \theta, \sin \theta, r \rangle
$$

$$
\mathbf{r}_\theta = \frac{\partial \mathbf{r}}{\partial \theta} = \langle -r \sin \theta, r \cos \theta, 0 \rangle
$$

The cross product:

$$
\mathbf{r}_r \times \mathbf{r}_\theta =
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\cos \theta & \sin \theta & r \\
- r \sin \theta & r \cos \theta & 0
\end{vmatrix}
$$

Calculating the determinant:

$$
\mathbf{r}_r \times \mathbf{r}_\theta = \left(- r^2 \cos \theta, - r^2 \sin \theta, r \right)
$$

Magnitude:

$$
|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{(- r^2 \cos \theta)^2 + (- r^2 \sin \theta)^2 + r^2}
$$

$$
= \sqrt{r^4 \cos^2 \theta + r^4 \sin^2 \theta + r^2}
$$

$$
= \sqrt{r^4 (\cos^2 \theta + \sin^2 \theta) + r^2}
$$

$$
= \sqrt{r^4 + r^2}
$$

$$
= r \sqrt{r^2 + 1}
$$

Thus, the surface area element:

$$
dS = |\mathbf{r}_r \times \mathbf{r}_\theta| dr d\theta = r \sqrt{r^2 + 1} dr d\theta
$$

Setting Up the Surface Area Integral

The total surface area depends on the specific region over which the paraboloid is being considered. For example, if the portion is bounded by a circle of radius R, then r varies from 0 to R, and θ from 0 to 2π.

The surface area S of the portion is:

$$
S = \int_0^{2\pi} \int_0^{R} r \sqrt{r^2 + 1} dr d\theta
$$

The integral over θ is straightforward:

$$
\int_0^{2\pi} d\theta = 2\pi
$$

The main challenge is evaluating:

$$
I = \int_0^{R} r \sqrt{r^2 + 1} dr
$$

Evaluating the Integral $I = \int r \sqrt{r^2 + 1} dr$

Method 1: Substitution

Let:

$$
u = r^2 + 1 \Rightarrow du = 2r dr \Rightarrow r dr = \frac{du}{2}
$$

Rewriting the integral:

$$
I = \int r \sqrt{r^2 + 1} dr = \frac{1}{2} \int \sqrt{u} du
$$

Calculate:

$$
\frac{1}{2} \int u^{1/2} du = \frac{1}{2} \times \frac{2}{3} u^{3/2} + C =
$$

$$
\frac{1}{3} u^{3/2} + C
$$

Back-substitute:

$$
u = r^2 + 1
$$

$$
I = \frac{1}{3} (r^2 + 1)^{3/2} + C
$$

Definite integral:

$$
\int_0^{R} r \sqrt{r^2 + 1} dr = \frac{1}{3} \left[(R^2 + 1)^{3/2} - 1^{3/2} \right] = \frac{1}{3} \left[(R^2 + 1)^{3/2} - 1 \right]
$$

Final Expression for the Surface Area

Putting it all together:

$$
S = 2\pi \times \frac{1}{3} \left[(R^2 + 1)^{3/2} - 1 \right]
$$

$$
S = \frac{2\pi}{3} \left[(R^2 + 1)^{3/2} - 1 \right]
$$

This formula gives the surface area of the portion of the paraboloid $2z = x^2 + y^2$ bounded by the circle $r = R$ in the xy-plane.

Special Cases and Applications

1. Entire paraboloid:
- When $R \to \infty$, the surface area becomes unbounded, indicating the paraboloid extends infinitely and has infinite surface area.

2. Bounded regions:
- For practical applications, such as calculating the surface area of a paraboloid segment, the radius R can be finite, representing a physical boundary like

Frequently Asked Questions

How do you express the surface area of the portion of the paraboloid $2z = x^2 + y^2$?

The surface area can be expressed using the surface integral formula: $A = \iint_D \sqrt{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2}\, dx\, dy$, where D is the projection of the surface onto the xy-plane.

What is the general approach to find the surface area of a paraboloid like $2z = x^2 + y^2$?

First, express z in terms of x and y, then compute the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$, substitute into the surface area formula, and evaluate the resulting double integral over the relevant domain.

How does changing the limits of the projection domain D affect the surface area calculation?

Changing the limits alters the portion of the paraboloid considered, thus affecting the size of the domain D. This, in turn, changes the limits of integration and the computed surface area accordingly.

What role do polar coordinates play in expressing the surface area of the paraboloid $2z = x^2 + y^2$?

Polar coordinates simplify the integration process by exploiting symmetry, allowing the double integral to be expressed in terms of r and θ, which can make the integral easier to evaluate, especially over circular domains.

Can you outline the key steps to derive the surface area formula for the paraboloid $2z = x^2 + y^2$?

Yes. First, rewrite $z = (x^2 + y^2)/2$; then find $\partial z/\partial x$ and $\partial z/\partial y$, compute the integrand $\sqrt{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2}$, determine the domain D, and evaluate the resulting double integral over D.

Additional Resources

Problem 2 [6 Marks; 3 Each] 2.1 Express The Surface Area Of The Portion Of The Paraboloid $2z = X^2 + Y^2$

Understanding the surface area of a three-dimensional shape is fundamental in fields ranging from engineering design to physical sciences. Today, we delve into a classic problem involving a paraboloid—a smooth, bowl-shaped surface—and focus on calculating its surface area over a specific region. This problem not only enhances comprehension of surface integrals but also demonstrates how calculus provides powerful tools to quantify complex geometrical objects.

Introduction to the Problem

The problem is succinct yet rich in mathematical implications: "Express the surface area of the portion of the paraboloid $2z = X^2 + Y^2$." At its core, this task involves determining the area of a curved surface defined by a quadratic relation between the coordinates. The challenge lies in translating a three-dimensional surface into a manageable mathematical expression, integrating over a specified region, and ultimately arriving at a formula that captures the surface's true extent.

This problem is typical in multivariable calculus, especially in the context of surface integrals, where the goal is to compute the area of a 3D surface expressed parametrically or explicitly. As we unfold the solution, we'll explore the necessary steps—parametrization, differential surface elements, and integration—while maintaining clarity and accessibility.

Understanding the Geometry: The Paraboloid $2z = x^2 + y^2$

Before jumping into calculations, it's essential to grasp the shape and properties of the paraboloid in question.

The Equation and Its Shape

The given surface is:

$$
2z = x^2 + y^2
$$

or equivalently,

$$
z = \frac{x^2 + y^2}{2}
$$

This describes a paraboloid opening upward, symmetric about the z-axis, with its vertex at the origin $(0, 0, 0)$. The quadratic relation indicates that as x and y grow in magnitude, the height z increases quadratically, resulting in a smooth, bowl-shaped surface.

The Portion of Interest

The problem asks for the surface area of a specific portion of this paraboloid. Typically, such problems specify a boundary—perhaps a circle at a certain height or radius. While the original question does not specify bounds explicitly, a common scenario involves calculating the surface area over a circular region in the xy-plane, say within a radius r.

For illustration, we may consider the portion over the disk:

$$
x^2 + y^2 \leq R^2
$$

which corresponds to the paraboloid between $z=0$ and $z=\frac{R^2}{2}$.

Understanding this helps simplify the problem since symmetry allows us to convert from Cartesian to polar coordinates seamlessly.

Parametrization of the Surface

Calculating surface area begins with parametrizing the surface—a process that transforms the complex surface into a manageable set of coordinates.

Choice of Coordinates

Given the circular symmetry, polar coordinates are the most natural choice:

$$
x = r \cos \theta,\quad y = r \sin \theta
$$

where:

- $r \geq 0$
- $\theta \in [0, 2\pi]$

This substitution simplifies the surface expression and the integration process.

Expressing z in Polar Coordinates

Substituting into the paraboloid equation:

$$
z = \frac{r^2}{2}
$$

Thus, the surface can be parametrized as:

$$
\mathbf{r}(r, \theta) = (r \cos \theta, r \sin \theta, \frac{r^2}{2})
$$

with $r \in [0, R]$ (or as per the problem's boundary) and $\theta \in [0, 2\pi]$.

This parametrization maps the two-dimensional domain in the $r-\theta$ plane onto the curved surface in 3D space.

Computing the Surface Element dS

The key to calculating surface area is understanding how small changes in r and θ translate into an infinitesimal patch of surface area.

Derivatives of the Parametrization

Calculate the partial derivatives:

$$
\frac{\partial \mathbf{r}}{\partial r} = (\cos \theta, \sin \theta, r)
$$

$$
\frac{\partial \mathbf{r}}{\partial \theta} = (-r \sin \theta, r \cos \theta, 0)
$$

These vectors span the tangent plane at each point on the surface, and their cross product yields a vector normal to the surface element.

Cross Product and Magnitude

Compute the cross product:

$$
\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} =
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\cos \theta & \sin \theta & r \\
- r \sin \theta & r \cos \theta & 0
\end{vmatrix}
$$

Calculating the determinant:

- $\mathbf{i}$ component:

$$
(\sin \theta)(0) - (r)(r \cos \theta) = - r^2 \cos \theta
$$

- $\mathbf{j}$ component:

$$
- [(\cos \theta)(0) - (r)(- r \sin \theta)] = - [0 + r^2 \sin \theta] = - r^2 \sin \theta
$$

- $\mathbf{k}$ component:

$$
(\cos \theta)(r \cos \theta) - (\sin \theta)(- r \sin \theta) = r (\cos^2 \theta + \sin^2 \theta) = r
$$

So, the cross product is:

$$
\mathbf{N} = (- r^2 \cos \theta, - r^2 \sin \theta, r)
$$

The magnitude of this vector gives the differential surface area element:

$$
|\mathbf{N}| = \sqrt{(- r^2 \cos \theta)^2 + (- r^2 \sin \theta)^2 + r^2}
$$

Simplify:

$$
|\mathbf{N}| = \sqrt{r^4 \cos^2 \theta + r^4 \sin^2 \theta + r^2}
$$

Since $\cos^2 \theta + \sin^2 \theta = 1$:

$$
|\mathbf{N}| = \sqrt{r^4 + r^2} = \sqrt{r^2 (r^2 + 1)} = r \sqrt{r^2 + 1}
$$

Setting Up the Surface Area Integral

The total surface area A over the region $0 \leq r \leq R$, $0 \leq \theta \leq 2\pi$, is:

$$
A = \int_{0}^{2\pi} \int_{0}^{R} |\mathbf{N}|\, dr\, d\theta
$$

Substituting the magnitude:

$$
A = \int_{0}^{2\pi} \int_{0}^{R} r \sqrt{r^2 + 1}\, dr\, d\theta
$$

The θ-dependence is straightforward:

$$
A = 2\pi \int_{0}^{R} r \sqrt{r^2 + 1}\, dr
$$

Now, focus on computing the single integral:

$$
I = \int_{0}^{R} r \sqrt{r^2 + 1}\, dr
$$

Calculating the Integral

This integral involves a typical substitution:

- Let $u = r^2 + 1$
- Then, $du = 2r\, dr$

Rearranged:

$$
r\, dr = \frac{1}{2} du
$$

Change limits:

- When $r = 0$, $u = 0^2 + 1 = 1$
- When $r = R$, $u = R^2 + 1$

Express the integral:

$$
I = \int_{u=1}^{u=R^2 + 1} \sqrt{u} \cdot \frac{1}{2} du = \frac{1}{2} \int_{1}^{R^2 + 1} u^{1/2} du
$$

Integrate:

$$
\frac{1}{2} \cdot \frac{2}{3} u^{3/2} \Big|_{1}^{R^2 + 1} = \frac{1}{3} \left[(R^2 + 1)^{3/2} - 1^{3/2}\right]
$$

Simplify:

$$
I = \frac{1}{3} \left[(R^2 + 1)^{3/2} - 1\right]
$$

Final Expression for Surface Area

Substituting back into the area integral:

\[

A = 2\pi \times \frac{1}{

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