Rewrite The Integral Rex2+y2dA, Where R Is The Semi-circular Region Bounded Below The X-axis And Above

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Introduction

Understanding how to evaluate integrals over specific regions is fundamental in multivariable calculus. This particular problem involves rewriting the integral of the function $R e x^2 + y^2 \, dA$, where R is a semi-circular region bounded below the x-axis and above, which implies that the region of integration is a semi-circle lying above the x-axis. To correctly approach this, we need to analyze the region, select an appropriate coordinate system, and express the integral accordingly.

Understanding the Region R

Description of the Semi-circular Region

- The region R is a semi-circle, which is half of a full circle.
- It is bounded below the x-axis and above, indicating it is the upper semi-circle.
- Typically, such a region is centered at the origin or some point with a specific radius.

Mathematical Representation of R

- Assuming the semi-circle is centered at the origin with radius r, the boundary is described by the circle:
$$
x^2 + y^2 = r^2
$$
- Since the region is above the x-axis:
$$
y \geq 0
$$
- The x-values range from $-r$ to r:
$$
-r \leq x \leq r
$$
- The y-values are determined by:
$$
y = \sqrt{r^2 - x^2}
$$

Choosing the Coordinate System

Cartesian Coordinates

- The region can be described directly using Cartesian coordinates with limits:
$$
x \in [-r, r], \quad y \in [0, \sqrt{r^2 - x^2}]
$$
- The integral becomes:
$$
\int_{x=-r}^{r} \int_{y=0}^{\sqrt{r^2 - x^2}} (R e x^2 + y^2) \, dy \, dx
$$
- However, integrating in Cartesian coordinates can be cumbersome because of the square root boundary.

Polar Coordinates

- Since the region is a semi-circle, polar coordinates are particularly convenient:
$$
x = r \cos \theta, \quad y = r \sin \theta
$$
- For the upper semi-circle:
$$
\theta \in [0, \pi]
$$
- The radius r varies from 0 to the circle's radius.

Transforming the Integral into Polar Coordinates

Expressing the Region R

- The region in polar coordinates:
$$
0 \leq r \leq R, \quad 0 \leq \theta \leq \pi
$$
- The differential area element in polar coordinates:
$$
dA = r \, dr \, d\theta
$$

Rewriting the Integrand

- The integrand $R e x^2 + y^2$ becomes:
$$
R e (r \cos \theta)^2 + (r \sin \theta)^2
$$

- Simplify:
$$
R e r^2 \cos^2 \theta + r^2 \sin^2 \theta
$$
- Factor r^2:
$$
r^2 (R e \cos^2 \theta + \sin^2 \theta)
$$

Formulating the Integral in Polar Coordinates

- The integral over the semi-circular region:
$$
\int_{\theta=0}^{\pi} \int_{r=0}^{R} r^2 (R e \cos^2 \theta + \sin^2 \theta) \, r \, dr \, d\theta
$$
- As $dA = r \, dr \, d\theta$:
$$
\int_{0}^{\pi} \int_{0}^{R} r^3 (R e \cos^2 \theta + \sin^2 \theta) \, dr \, d\theta
$$

Evaluating the Integral

Integral with Respect to r

- Compute:
$$
\int_{0}^{R} r^3 \, dr = \frac{R^4}{4}
$$
- The integral becomes:
$$
\frac{R^4}{4} \int_{0}^{\pi} (R e \cos^2 \theta + \sin^2 \theta) \, d\theta
$$

Integral with Respect to θ

- Break into two parts:
$$
R e \int_{0}^{\pi} \cos^2 \theta \, d\theta + \int_{0}^{\pi} \sin^2 \theta \, d\theta
$$
- Recall the standard integrals:
$$
\int_{0}^{\pi} \cos^2 \theta \, d\theta = \frac{\pi}{2}
$$
$$
\int_{0}^{\pi} \sin^2 \theta \, d\theta = \frac{\pi}{2}
$$

- Therefore:
$$
R e \cdot \frac{\pi}{2} + \frac{\pi}{2} = \frac{\pi}{2} (R e + 1)
$$

Final Expression of the Rewritten Integral

- Combining all parts:
$$
\boxed{
\frac{R^4}{4} \times \frac{\pi}{2} (R e + 1) = \frac{\pi R^4}{8} (R e + 1)
}
$$
- This is the value of the original integral over the semi-circular region R.

Summary and Additional Notes

- Key steps involved:
1. Identifying the region R as a semi-circle above the x-axis.
2. Choosing an appropriate coordinate system (polar coordinates) for simplicity.
3. Converting the integrand into polar form.
4. Setting the proper limits for r and θ.
5. Performing the integrations sequentially.
- Extensions:
- If the radius R of the semi-circle is not specified, the integral can be expressed in terms of R.
- For more complex regions, other coordinate systems or transformations may be required.
- The methodology exemplified here illustrates general strategies for evaluating integrals over curved regions.

Conclusion

Rewriting the integral $R e x^2 + y^2 \, dA$ over a semi-circular region involves understanding the geometry of the region, selecting an optimal coordinate system, and carefully transforming both the limits and the integrand. Polar coordinates often provide a straightforward approach for circular and semi-circular regions due to their natural alignment with radial symmetry. Through this process, the integral simplifies into manageable parts, enabling exact evaluation. Mastery of such techniques is essential for tackling a broad range of problems in multivariable calculus, physics, and engineering disciplines where integration over curved regions is common.

Frequently Asked Questions

What is the geometric region R in the integral Rex2 + y2 dA?

Region R is the semi-circular area bounded above by the semicircle ($y = \sqrt{r^2 - x^2}$) and below by the x-axis, typically with the semicircle centered at the origin.

How do you set up the integral over the semi-circular region R?

You can set up the integral using polar coordinates, where $x = r \cos \theta$, $y = r \sin \theta$, with θ from π to 2π and r from 0 to the radius of the semicircle.

What is the value of the integral Rex2 + y2 dA over the semi-circular region R?

The value depends on the radius of the semicircle; typically, converting to polar coordinates simplifies the integral to a form involving the radius r and angle θ, which can then be integrated accordingly.

How do polar coordinates simplify the integration over the semi-circular region R?

In polar coordinates, the region R becomes a sector of a circle, making the limits straightforward: r from 0 to R and θ from π to 2π, and the integrand becomes $r^2 (\cos^2 \theta + \sin^2 \theta) = r^2$.

What is the significance of the bounds on θ in the integral over R?

The bounds on θ from π to 2π represent the semi-circular region below the x-axis, covering the lower half of the circle.

Can the integral Rex2 + y2 dA be computed directly in Cartesian coordinates?

While possible, it is more complex; using polar coordinates is generally preferred for circular regions due to simpler limits and integrand expressions.

What is the general formula for converting a double integral over a circular region to polar coordinates?

The double integral over a circular region R becomes $\iint_R f(x,y)\, dA = \int_{\theta=\theta_1}^{\theta_2} \int_{r=0}^{R} f(r \cos \theta, r \sin \theta)\, r\, dr\, d\theta$.

How does the symmetry of the semi-circular region R affect the integral of Rex2 + y2 dA?

Due to symmetry, the integral of x^2 and y^2 over the semi-circle can sometimes be related or simplified, especially since x^2 and y^2 are symmetric functions over the semi-circular domain.

What are common challenges when evaluating integrals over semi-circular regions?

Challenges include correctly setting the limits for r and θ, converting the integrand into the appropriate coordinate system, and accounting for the region's boundaries accurately.

How does the radius of the semi-circular region influence the value of the integral Rex2 + y2 dA?

The radius R determines the upper limit of r in polar coordinates; as R increases, the value of the integral increases proportionally, often involving R^4 terms when integrating r^2 over the circle.

Additional Resources

Rewrite The Integral Rex2 + y² dA, Where R Is The Semi-circular Region Bounded Below The X-axis And Above

Understanding how to evaluate integrals over specific regions is a fundamental skill in multivariable calculus. When working with regions such as semi-circular areas, converting the integral into a more manageable form—often through coordinate transformations—can significantly simplify the process. In this article, we will explore how to rewrite the integral Rex² + y² dA, where R is the semi-circular region bounded below the x-axis and above, providing a comprehensive guide that includes the problem's geometric context, coordinate system choices, and step-by-step integration techniques.

The Geometric Context of the Region R

Before diving into the integral itself, it’s crucial to understand the geometric nature of the region R. Specifically, the problem states that R is a semi-circular region bounded below the x-axis and above, which suggests the following:

- R is a semi-circle with a certain radius, say r.
- The semi-circle is located above the x-axis, meaning it occupies the upper half of the circle.
- It is bounded below the x-axis, which marks the diameter of the semi-circle.

Visualizing the region:

- Imagine a circle centered at the origin, with radius r, described by the equation: $x^2 + y^2 = r^2$.
- The semi-circular region R is the upper half of this circle, where $y \geq 0$, but since the problem states it is bounded below the x-axis, it actually refers to the lower half of the circle, i.e., where $y \leq 0$.
- To clarify, if the region is bounded below the x-axis and above the circle, then R is the lower semi-circular region with $y \leq 0$, bounded by $x^2 + y^2 = r^2$ and $y \leq 0$.

Summary:

- Region R: The semi-circle defined by $x^2 + y^2 \leq r^2$, with $y \leq 0$ (lower semi-circular region).

Choosing the Coordinate System: Cartesian vs. Polar

When integrating over a circular region, polar coordinates often provide an elegant and efficient approach. Let’s explore the options:

Cartesian Coordinates

- The integral in Cartesian coordinates is written as:

$$
\iint_{R} (x^2 + y^2) \, dA
$$

- The domain R in Cartesian coordinates is defined by inequalities:

$$
x^2 + y^2 \leq r^2, \quad y \leq 0
$$

- To evaluate this integral directly, one would set up the bounds as:

$$
x \text{ from } -r \text{ to } r
$$

$$
y \text{ from } -\sqrt{r^2 - x^2} \text{ to } 0
$$

- The limits are straightforward but can lead to more complex integration due to the square root bounds.

Polar Coordinates

- Polar coordinates are defined by:

\[

$$
x = r \cos \theta, \quad y = r \sin \theta
$$

- The Jacobian determinant (area element transformation) is:

$$
dA = r \, dr \, d\theta
$$

- Region R in polar coordinates:

- Since R is the lower semi-circle, the angular bounds are:

$$
\theta \text{ from } \pi \text{ to } 2\pi
$$

- The radial bounds are:

$$
r \text{ from } 0 \text{ to } r
$$

- The integrand becomes:

$$
x^2 + y^2 = r^2
$$

- Advantages of polar coordinates:

- Simplifies the integrand to a function of r only.
- The bounds are straightforward, often leading to easier integrations.

Given these considerations, polar coordinates are typically preferred for circular regions.

Setting Up the Integral in Polar Coordinates

Step 1: Express the integrand

$$
x^{2} + y^{2} = r^{2}
$$

Step 2: Determine the bounds

- Angular bounds:

- Since the region is the lower semi-circle, the angle θ spans from π to 2π.

- Radial bounds:

- For the full circle of radius r:

$$
0 \leq r \leq R
$$

Step 3: Write the integral

$$
\iint_{R} (x^{2} + y^{2}) dA = \int_{\theta=\pi}^{2\pi} \int_{r=0}^{R} r^{2} \cdot r \, dr \, d\theta
$$

which simplifies to:

$$
\int_{\pi}^{2\pi} \int_{0}^{R} r^{3} \, dr \, d\theta
$$

Evaluating the Integral

Step 1: Integrate with respect to r

$$
\int_{0}^{R} r^{3} \, dr = \frac{r^{4}}{4} \bigg|_{0}^{R} = \frac{R^{4}}{4}
$$

Step 2: Integrate with respect to θ

$$
\int_{\pi}^{2\pi} d\theta = 2\pi - \pi = \pi
$$

Final result:

$$
\boxed{
\iint_{R} (x^{2} + y^{2}) dA = \pi \cdot \frac{R^{4}}{4} = \frac{\pi R^{4}}{4}
}
$$

This is the value of the integral over the lower semi-circular region of radius R.

Generalizing for Different Semi-circular Regions

The above process assumes the radius R is given explicitly. For a more general scenario, or if the region is defined differently, the same approach applies with modifications:

- Adjust the angular bounds to match the region's location (e.g., upper semi-circle: θ from 0 to π)
- If the circle is centered elsewhere, shift the coordinate system accordingly

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Additional Tips and Considerations

- Symmetry:

Recognizing symmetry can greatly simplify the calculation. For instance, if the integrand is radially symmetric (like $x^{2} + y^{2}$), integrating over half the circle and then doubling the result can save effort.

- Changing the order of integration:

At times, switching the order (e.g., integrating over θ first or r first) may be advantageous, particularly when bounds are complicated.

- Alternative coordinate systems:

If the region R is not a perfect circle but an elliptical or other shape, consider elliptical coordinates or other transformations.

- Visualization:

Drawing the region helps clarify bounds and ensures the correct limits are used. Use software tools or graphing calculators for visualization if needed.

Summary: Key Steps to Rewrite and Evaluate the Integral

1. Identify the region R:

- Understand the geometric shape and bounds.
- For a semi-circular region bounded below the x-axis and above, recognize it as a semi-circle of radius R.

2. Choose an appropriate coordinate system:

- Polar coordinates are often best for circular regions.

3. Express the integrand in the chosen coordinates:

- For $x^{2} + y^{2}$, it simplifies to r^{2}.

4. Determine the bounds:

- Angular bounds: typically from π to 2π for the lower semi-circle.
- Radial bounds: from 0 to the radius R.

5. Set up the integral with the Jacobian:

$$
\iint_{R} (x^{2} + y^{2}) dA = \int_{\theta} \int_{r} r^{2} \cdot r \, dr \, d\theta
$$

6. Perform integration step-by-step:

- Integrate with respect to r, then θ.

7. Interpret the result:

- The final expression provides the value of the integral over the specified semi-circular region.

Final Remarks

Rewriting integrals over geometric regions like semi-circles requires understanding both the shape's symmetry and the most suitable coordinate system. Polar coordinates are particularly powerful for circular and semi-circular areas, transforming complex bounds into straightforward intervals. Whether calculating moments, areas, or other physical quantities, mastering these transformations enhances both the efficiency and accuracy of your integrations.

By following this structured approach—identifying the region, choosing the proper coordinate system, setting bounds carefully, and executing the integration systematically—you can confidently evaluate integrals over semi-circular and other curved regions.

Rewrite The Integral Rex2 Y2da Where R Is The Semi Circular Region Bounded Below The X Axis And Above

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