Set Up A Double Integral For Calculating The Flux Of F = 5xi + Yj + Zk Through The Part Of The Surface

Set Up A Double Integral For Calculating The Flux Of F = 5xi + Yj + Zk Through The Part Of The Surface is a fundamental problem in vector calculus, particularly within the context of flux integrals and surface integrals. Calculating the flux of a vector field across a surface involves understanding how the field passes through a given surface, and setting up the appropriate double integral is a crucial step in this process. Properly establishing this integral requires identifying the surface, parametrizing it, determining the surface element, and applying the surface integral formula. This article provides a comprehensive guide to setting up a double integral for calculating flux, focusing on the vector field F = 5xi + yj + zk and the relevant surface.

Understanding the Concept of Flux in Vector Calculus

Flux is a measure of how much of a vector field passes through a surface. In mathematical terms, the flux of a vector field F across a surface S is given by the surface integral:

$$
\Phi = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS
$$

where:

- F is the vector field.
- n is the unit normal vector to surface S.
- dS is an infinitesimal element of surface area.

The goal is to evaluate this integral, which often involves converting the surface integral into a double integral over a parameter domain.

Step-by-Step Guide to Setting Up the Double Integral for Flux Calculation

To accurately set up the double integral, follow these systematic steps:

1. Identify and Describe the Surface

- Determine the surface S through which flux is to be calculated.
- Understand its geometric nature (plane, sphere, paraboloid, etc.).
- Find a suitable parametrization that covers S.

2. Choose a Suitable Parameterization

- Express the surface S in terms of two parameters, typically u and v.
- The parameterization should be smooth and cover the part of the surface of interest.

3. Compute the Surface Element $d\mathbf{S}$

- Find the surface differential element, which involves calculating the cross product of partial derivatives of the position vector.
- The differential surface element $d\mathbf{S}$ is given by:

$$
d\mathbf{S} = (\mathbf{r}_u \times \mathbf{r}_v) \, du \, dv
$$

where $\mathbf{r}_u$ and $\mathbf{r}_v$ are the partial derivatives of the position vector $\mathbf{r}(u, v)$.

4. Express the Vector Field $\mathbf{F}$ in Terms of Parameters

- Substitute the parameterization into F to express it as $\mathbf{F}(u, v)$.

5. Compute the Dot Product $\mathbf{F} \cdot d\mathbf{S}$

- Take the dot product of the vector field $\mathbf{F}(u, v)$ with the surface element $d\mathbf{S}$.

6. Set Up the Double Integral

- Integrate over the domain D in the uv-plane:

\[

$$
\Phi = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, du \, dv
$$

Applying the Method to the Given Vector Field and Surface

Let's contextualize these steps with the specific vector field F = 5xi + yj + zk and the given surface.

Example Scenario

Suppose we are asked to compute the flux of F = 5xi + yj + zk through a specific part of a surface, for example, a portion of the paraboloid $z = x^2 + y^2$ over a disk $x^2 + y^2 \leq R^2$.

Step-by-Step Example: Flux Through a Paraboloid Surface

1. Define the Surface

- Surface: $z = x^2 + y^2$
- Region: $x^2 + y^2 \leq R^2$

2. Parameterization of the Surface

- Use polar coordinates:

$$
x = r \cos \theta, \quad y = r \sin \theta, \quad z = r^2
$$

where:

- $r \in [0, R]$

- $\theta \in [0, 2\pi]$

- The position vector:

$$
\mathbf{r}(r, \theta) = \langle r \cos \theta, r \sin \theta, r^2 \rangle
$$

3. Compute the Surface Element $d\mathbf{S}$

- Find partial derivatives:

$$
\mathbf{r}_r = \frac{\partial \mathbf{r}}{\partial r} = \langle \cos \theta, \sin \theta, 2r \rangle
$$

$$
\mathbf{r}_\theta = \frac{\partial \mathbf{r}}{\partial \theta} = \langle -r \sin \theta, r \cos \theta, 0 \rangle
$$

- Compute the cross product:

$$
\mathbf{r}_r \times \mathbf{r}_\theta =
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\cos \theta & \sin \theta & 2r \\
- r \sin \theta & r \cos \theta & 0
\end{vmatrix}
$$

Calculating the determinant:

$$
\mathbf{r}_r \times \mathbf{r}_\theta = \left| \begin{aligned}
& (\sin \theta)(0) - (2r)(r \cos \theta) \\
- \left[(\cos \theta)(0) - (2r)(- r \sin \theta) \right] \\
(\cos \theta)(r \cos \theta) - (\sin \theta)(- r \sin \theta)
\end{aligned} \right|
$$

Simplifies to:

$$
\mathbf{r}_r \times \mathbf{r}_\theta = \langle -2 r^2 \cos \theta, - 2 r^2 \sin \theta, r (\cos^2 \theta + \sin^2 \theta) \rangle
$$

Since $\cos^2 \theta + \sin^2 \theta = 1$:

$$
\mathbf{r}_r \times \mathbf{r}_\theta = \langle -2 r^2 \cos \theta, - 2 r^2 \sin \theta, r \rangle
$$

- The magnitude of this vector gives the surface element:

$$
d\mathbf{S} = \mathbf{r}_r \times \mathbf{r}_\theta \, dr \, d\theta
$$

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4. Express $\mathbf{F}$ in Terms of r and θ

- Recall $x = r \cos \theta$, $y = r \sin \theta$, $z = r^2$.

- Thus:

$$
\mathbf{F} = 5 x \mathbf{i} + y \mathbf{j} + z \mathbf{k} = 5 r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + r^2 \mathbf{k}
$$

---

5. Compute $\mathbf{F} \cdot (\mathbf{r}_r \times \mathbf{r}_\theta)$

- Dot product:

$$
\mathbf{F} \cdot (\mathbf{r}_r \times \mathbf{r}_\theta) =
\left(5 r \cos \theta \right)(-2 r^2 \cos \theta) + \left(r \sin \theta \right)(-2 r^2 \sin \theta) + r^2 (r)
$$

Simplify:

$$
= -10 r^3 \cos^2 \theta - 2 r^3 \sin^2 \theta + r^3
$$

Combine terms:

$$
= r^3 \left(-10 \cos^2 \theta - 2 \sin^2 \theta + 1 \right)
$$

---

6. Set Up the Double Integral

- The flux Φ over the region:

$$
\Phi = \int_0^{2\pi} \int_0^R \mathbf{F} \cdot (\mathbf{r}_r \times \mathbf{r}_\theta) \, dr \, d\theta
$$

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Frequently Asked Questions

How do I identify the surface for setting up the double integral when calculating flux of F = 5xi + yj + zk?

You need to determine the specific part of the surface through which the flux is being calculated, such as a plane, paraboloid, or other surface, and express it parametrically or explicitly in terms of variables like x and y.

What is the general approach to setting up a double integral for flux calculation of the vector field F = 5xi + yj + zk?

First, find the surface's parameterization or an equation describing it, compute the surface element dS, and then evaluate the surface integral of F · n dS, which reduces to a double integral over the projected domain in the xy-plane.

How do I determine the limits of integration for the double integral when calculating flux through a surface?

The limits are determined by the projection of the surface onto the xy-plane (or another suitable plane). You identify the region R in the xy-plane over which the surface extends, and set the integration bounds accordingly.

What role does the surface normal vector play in setting up the double integral for flux?

The surface normal vector n is used to compute the surface element $dS = n\,dS$, and the flux integral involves the dot product $F \cdot n$. When setting up the double integral, you often express dS in terms of dx and dy using the surface parameterization.

Can you give an example of setting up the double integral for flux through a plane surface for the vector field F = 5xi + yj + zk?

Suppose the surface is part of the plane $z = 2 - x - y$ over a region R in the xy-plane. The double integral for flux would be ∫∫_{R} $F \cdot n\,dS$, where n is the surface normal, often expressed as
abla g / |
abla g| if $g(x,y,z)=0$ describes the surface, and then evaluated over R.

What are common mistakes to avoid when setting up the double integral for flux through a surface?

Common mistakes include incorrect identification of the surface limits, forgetting to include the surface's orientation (normal direction), neglecting the Jacobian when parameterizing the surface, and mixing up the order of integration bounds.

How does the divergence theorem relate to setting up a double integral for flux calculations?

The divergence theorem links the flux integral over a closed surface to a triple integral of the divergence over the volume enclosed. While it simplifies some calculations, setting up the double integral directly involves parameterizing the surface and integrating $F \cdot n\,dS$.

What steps should I follow to systematically set up the double integral for calculating flux of F = 5xi + yj + zk through a given surface?

First, parametrize or find an explicit form of the surface. Second, determine the surface normal vector and dS. Third, identify the projection region in the xy-plane to set integration limits. Fourth, compute the dot product $F \cdot n$, and finally, write the double integral over the region with the integrand as $F \cdot n\,dS$.

Additional Resources

Set Up A Double Integral For Calculating The Flux Of $F = 5xi + yj + zk$ Through The Part Of The Surface

Calculating flux through a surface is a fundamental concept in vector calculus, with applications spanning physics, engineering, and mathematical analysis. Specifically, when working with a vector field such as F = 5xi + yj + zk, understanding how to properly set up a double integral to evaluate the flux through a particular surface is essential. This detailed guide will walk you through the process, emphasizing clarity, methodology, and the nuances involved in the setup.

Understanding the Concept of Flux in Vector Fields

Before diving into the setup process, it’s crucial to understand what flux represents in the context of vector fields:

- Flux quantifies the amount of the vector field passing through a given surface.
- Mathematically, the flux Φ of a vector field F across a surface S is given by the surface integral:

$$
\Phi = \iint_{S} \mathbf{F} \cdot \mathbf{n}\, dS
$$

where:
- $\mathbf{F}$ is the vector field, in this case, $\mathbf{F} = 5x\, \mathbf{i} + y\, \mathbf{j} + z\, \mathbf{k}$.
- $\mathbf{n}$ is the unit normal vector to the surface S.
- dS is the differential surface element.

- The dot product $\mathbf{F} \cdot \mathbf{n}$ indicates the component of F passing through the surface in the direction of the surface normal.

Key Steps to Set Up the Double Integral

The process involves several systematic steps:

1. Identify the Surface S

- Determine the geometric description of the surface through which flux is being calculated.
- The surface could be part of a plane, a curved surface, or a more complex shape.
- Clarify the bounds and the orientation (which side is "inside" vs. "outside") to ensure correct normal vector direction.

2. Parameterize the Surface

- Express the surface using two parameters, typically u and v, leading to a vector function $\mathbf{r}(u, v)$.
- This parameterization simplifies the surface integral, allowing the calculation of $d\mathbf{S}$.

3. Determine the Surface Element $d\mathbf{S}$

- Compute the cross product of the partial derivatives $\mathbf{r}_u$ and $\mathbf{r}_v$, which yields the oriented surface element vector:

$$
d\mathbf{S} = (\mathbf{r}_u \times \mathbf{r}_v) du\, dv
$$

- The magnitude $|\mathbf{r}_u \times \mathbf{r}_v|$ gives the differential surface area dS.

4. Express the Vector Field $\mathbf{F}$ in Terms of Parameters

- Substitute the parameterization into $\mathbf{F}$ to write it as $\mathbf{F}(u, v)$.

5. Set Up the Surface Integral

- Write the flux as:

$$
\Phi = \iint_{D} \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) du\, dv
$$

- Here, D is the domain of the parameters u, v, derived from the geometric boundaries of the surface.

Applying the Process to a Specific Surface

Suppose you are given a surface S such as a portion of the plane or a curved surface. Here's how to proceed:

Example: Part of a Plane

Imagine the surface S is part of the plane $z = 2x + y$, bounded over a specific region in the xy-plane, say a rectangle:

- Region R: x from a to b, y from c to d.

Step-by-step setup:

1. Parameterization:

- Use x and y as parameters:

$$
\mathbf{r}(x, y) = \langle x, y, 2x + y \rangle
$$

2. Compute $\mathbf{r}_x$ and $\mathbf{r}_y$:

$$
\mathbf{r}_x = \frac{\partial \mathbf{r}}{\partial x} = \langle 1, 0, 2 \rangle
$$

$$
\mathbf{r}_y = \frac{\partial \mathbf{r}}{\partial y} = \langle 0, 1, 1 \rangle
$$

3. Calculate $\mathbf{r}_x \times \mathbf{r}_y$:

$$
\mathbf{r}_x \times \mathbf{r}_y =
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
1 & 0 & 2 \\
0 & 1 & 1
\end{vmatrix}
= \mathbf{i}(0 \times 1 - 2 \times 1) - \mathbf{j}(1 \times 1 - 2 \times 0) + \mathbf{k}(1 \times 1 - 0 \times 0)
$$

$$
= \mathbf{i}(0 - 2) - \mathbf{j}(1 - 0) + \mathbf{k}(1 - 0)
= -2 \mathbf{i} - 1 \mathbf{j} + 1 \mathbf{k}
$$

4. Express $\mathbf{F}$ on the surface:

$$
\mathbf{F}(x, y) = 5x\, \mathbf{i} + y\, \mathbf{j} + (2x + y)\, \mathbf{k}
$$

5. Set up the flux integral:

$$
\Phi = \iint_{R} \mathbf{F}(x, y) \cdot (\mathbf{r}_x \times \mathbf{r}_y) dx\, dy
$$

Substituting:

$$
\mathbf{F}(x, y) \cdot (\mathbf{r}_x \times \mathbf{r}_y) = (5x, y, 2x + y) \cdot (-2, -1, 1)
$$

Compute the dot product:

$$
= 5x(-2) + y(-1) + (2x + y)(1) = -10x - y + 2x + y = -8x
$$

Notice how the y-terms cancel out, simplifying the integrand.

6. Final integral:

$$
\Phi = \iint_{R} -8x\, dx\, dy
$$

- The bounds of x and y are determined by the region R.

Specifying the Domain of Integration

The domain D depends on the part of the surface under consideration:

- For a rectangular region, bounds are straightforward.
- For curved or irregular regions, you may need to express bounds in terms of inequalities or switch to polar or other coordinate systems.

Example:

- If R is the rectangle $a \leq x \leq b$, $c \leq y \leq d$, then:

$$
\Phi = \int_{x=a}^{b} \int_{y=c}^{d} -8x\, dy\, dx
$$

- Alternatively, if the region is more complex, define the bounds accordingly, e.g., in polar coordinates or using inequalities.

Choosing the Proper Orientation and Normal

Vector

The direction of the surface normal vector \(\mathbf{n}\) influences the sign of the flux:

- Typically, the surface parameterization and the cross product \(\mathbf{r}_u \times \mathbf{r}_v\) produce an oriented surface element.
- Confirm that the chosen orientation aligns with the physical or problem-specific context.
- If the normal vector points inward or outward, adjust accordingly, possibly by multiplying the integrand by \(-1\).

Generalization to Other Surfaces

While the above example illustrates a plane, the methodology extends to:

- Curved Surfaces: Use appropriate parameterizations (for spheres, cylinders, paraboloids, etc.).
- Surfaces with Boundaries: Carefully analyze the boundary curves to set bounds.
- Parametric Surfaces in Different Coordinates: Sometimes, switching to cylindrical or spherical coordinates simplifies the setup.

Additional Considerations and Tips

- Simplify the integrand: Always attempt to simplify the dot product before integrating

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