Show That Any Linear Association Of Sinct And Coswt, Such That X(t) = A Coswt + A Sinut, With Constant

Show That Any Linear Association Of Sinct And Coswt, Such That X(t) = A Coswt + A Sinut, With Constant is a fundamental concept in signal processing and harmonic analysis. This statement explores the relationship between sinusoidal functions and their linear combinations, revealing how such combinations can be expressed in terms of a single sinusoid with a phase shift. Understanding this principle is crucial in various engineering disciplines, especially in the analysis and synthesis of signals, Fourier analysis, and system responses. In this article, we will examine this association thoroughly, providing mathematical proofs, interpretations, and practical applications.

Introduction to Sinusoidal Functions and Linear Combinations

Sinusoidal functions, specifically sine and cosine functions, are the building blocks of many periodic signals. They are characterized by their amplitude, frequency, and phase. A general sinusoid can be written as:
- $\cos \omega t$
- $\sin \omega t$

where ω is the angular frequency, and t is time.

A linear combination of these functions takes the form:
$$X(t) = A \cos \omega t + B \sin \omega t$$
where A and B are constants, representing the coefficients of the cosine and sine components, respectively.

The question arises: can this linear combination be rewritten as a single sinusoid with some amplitude and phase? The answer is yes, and this process involves understanding the phase shift and amplitude modulation.

Expressing a Linear Combination of Sinusoidals as a Single Sinusoid

The key insight is that any linear combination of sine and cosine functions at the same frequency can be expressed as a single sinusoid:
$$X(t) = R \cos (\omega t - \phi)$$
or equivalently,
$$X(t) = R \cos \omega t \cos \phi + R \sin \omega t \sin \phi$$
which, upon comparison, suggests the relationships:
- $R \cos \phi = A$
- $R \sin \phi = B$

This is a classic trigonometric identity that allows us to combine the two coefficients into a single amplitude and phase shift.

Mathematical Derivation

Let’s derive the expression step by step:

Step 1: Starting with the Linear Combination

$$ X(t) = A \cos \omega t + B \sin \omega t $$

Step 2: Express as a Single Cosine Function

Assuming:
$$ X(t) = R \cos (\omega t - \phi) $$
we expand the right side using the cosine subtraction formula:
$$ R \cos (\omega t - \phi) = R \left(\cos \omega t \cos \phi + \sin \omega t \sin \phi \right) $$
which simplifies to:
$$ X(t) = R \cos \phi \cos \omega t + R \sin \phi \sin \omega t $$

Step 3: Equate Coefficients

Matching coefficients:
$$
\begin{cases}
A = R \cos \phi \\
B = R \sin \phi
\end{cases}
$$

Step 4: Solve for R and ϕ

The amplitude R can be found using the Pythagorean theorem:
$$ R = \sqrt{A^2 + B^2} $$
and the phase ϕ is given by:
$$ \phi = \arctan \left(\frac{B}{A} \right) $$
(considering the correct quadrant based on the signs of A and B).

Implications of the Transformation

This derivation confirms that any linear combination of sine and cosine for the same frequency can be represented as a single sinusoid with an appropriately chosen amplitude R and phase shift ϕ. This has several important implications:

- **Simplification of Signal Analysis:** Complex signals composed of multiple sinusoidal components can be simplified into a single sinusoid, easing analysis and interpretation.
- **System Response Characterization:** In linear systems, the output response to sinusoidal inputs can be expressed in terms of amplitude and phase shift, facilitating system design and control.
- **Fourier Series and Transforms:** The principle underpins the Fourier analysis, allowing signals to be decomposed into sinusoidal components and reconstructed efficiently.

Application in Signal Processing

The ability to express a linear combination of sinusoidals as a single sinusoid is widely used in practical scenarios:

Amplitude Modulation (AM)

In communication systems, signals are often modulated by varying amplitude, which involves combining a carrier sinusoid with a message signal. Recognizing that the combined signal can be expressed as a shifted sinusoid simplifies demodulation and filtering processes.

Filtering and Signal Synthesis

Filters are designed to emphasize or suppress certain frequencies. When signals are represented as single sinusoids, designing filters becomes more straightforward, and the phase relationships are easier to control.

Harmonic Analysis in Mechanical Systems

Mechanical vibrations often involve multiple sinusoidal components due to various harmonics. Understanding their linear combinations allows engineers to identify dominant frequencies and mitigate unwanted vibrations.

Extending to Multiple Frequencies

While the discussion so far has focused on a single frequency, real-world signals often comprise multiple frequencies. The principle extends to the superposition of multiple sinusoids:
\[X(t) = \sum_{k=1}^{n} \left(A_k \cos \omega_k t + B_k \sin \omega_k t \right) \]
which can be viewed as a sum of individual sinusoidal components, each expressible as a single sinusoid. The superposition principle holds, and the overall signal can be analyzed in terms of its constituent amplitudes and phase shifts.

Conclusion

In summary, the statement that any linear association of $\sin \omega t$ and $\cos \omega t$ can be expressed as a single sinusoid with constant amplitude and phase shift is a fundamental concept in harmonic analysis and signal processing. The mathematical derivation confirms that:

$$A \cos \omega t + B \sin \omega t = R \cos (\omega t - \phi)$$

where:

$$
\begin{cases}
R = \sqrt{A^2 + B^2} \\
\phi = \arctan \left(\frac{B}{A} \right)
\end{cases}
$$

This transformation simplifies the analysis and synthesis of signals and underpins many modern communication and control systems. Understanding this relationship enhances our ability to manipulate sinusoidal signals effectively, ensuring accurate modeling, filtering, and system design across various engineering disciplines.

Frequently Asked Questions

What is the general form of a linear association involving sine and cosine functions for a given signal?

A general linear association involves expressing the signal as a linear combination of sine and cosine functions, such as $X(t) = A \cos(\omega t) + B \sin(\omega t)$, where A and B are constants.

How can we show that the sum of A cos(ωt) and B sin(ωt) can be represented as a single sinusoid?

By using the identity: $A \cos(\omega t) + B \sin(\omega t) = R \cos(\omega t - \varphi)$, where $R = \sqrt{(A^2 + B^2)}$ and φ = arctangent(B/A), thus representing the sum as a single sinusoid with amplitude R and phase φ.

What are the conditions for the constants A and B in the linear association X(t) = A cos(ωt) + B sin(ωt) to ensure a valid model?

The constants A and B are arbitrary real numbers that define the amplitude and phase of the sinusoid; their values are typically determined by initial conditions or data fitting, with no specific restrictions other than being real numbers.

How does the linear association of sine and cosine functions relate to harmonic oscillations?

Linear combinations of sine and cosine functions model harmonic oscillations, representing oscillatory behavior with specific amplitude and phase, which are fundamental in analyzing periodic signals.

What is the significance of representing X(t) as A cos(ωt) + B sin(ωt) with constant coefficients?

It simplifies the analysis of periodic signals, allows for easy computation of amplitude and phase, and facilitates understanding of the signal's properties such as energy and power.

Can any linear combination of sine and cosine functions with constant coefficients be expressed as a single sinusoid? If so, how?

Yes. Any linear combination A cos(ωt) + B sin(ωt) can be expressed as R cos(ωt - φ), where $R = \sqrt{(A^2 + B^2)}$ and φ = arctangent(B/A), consolidating the sum into a single sinusoidal function.

What mathematical tool is used to combine A cos(ωt) + B sin(ωt) into a single sinusoidal function?

The use of the amplitude-phase transformation or phasor representation, employing the Pythagorean theorem and phase angle calculation, is used to combine the terms into a single sinusoid.

How does the phase shift φ relate to the constants A and B in the linear combination?

The phase shift φ is given by φ = arctangent(B/A), indicating the angle by which the sinusoid is shifted relative to the cosine function.

Why is expressing a linear association of sine and cosine functions as a single sinusoid useful in signal processing?

It simplifies analysis, filtering, and interpretation of signals, making it easier to understand amplitude, phase, and frequency components, which are fundamental in Fourier analysis and system response analysis.

What is the main theorem or principle that justifies representing a linear combination of sine and cosine as a single sinusoid?

The principle is based on the superposition of sinusoidal functions and the use of trigonometric identities, specifically the cosine difference identity, which allows rewriting the sum as a single sinusoid with adjusted amplitude and phase.

Additional Resources

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Introduction

In the realm of signal processing, harmonic analysis, and systems theory, the relationship between sinusoidal functions and their linear combinations forms a foundational concept. The specific form $X(t) = A \cos \omega t + A \sin \omega t$, where A is a constant amplitude and ω is the angular frequency, often appears in the analysis of periodic signals, Fourier series, and differential equations. Establishing that such a combination can be expressed as a single sinusoid with a phase shift underscores the elegant structure of sinusoidal functions and their linear combinations.

This investigation aims to rigorously demonstrate that any linear association of $\sin \omega t$ and $\cos \omega t$ of the form $X(t) = A \cos \omega t + A \sin \omega t$, with constants, can be represented as a single sinusoid with a phase shift. The significance extends beyond theoretical elegance; it facilitates simplifying complex signal representations, analyzing system responses, and designing filters.

Mathematical Foundation

The Basic Sinusoidal Functions

The fundamental sinusoidal functions, sine and cosine, are orthogonal over specific intervals and possess properties that make them ideal basis functions in Fourier analysis. They are defined as:

- $\sin \omega t$: sine wave with angular frequency ω
- $\cos \omega t$: cosine wave with angular frequency ω

These functions are periodic with period $T = \frac{2\pi}{\omega}$ and are related through phase shifts:

$$
\sin \omega t = \cos \left(\omega t - \frac{\pi}{2} \right)
$$

The General Linear Combination

Formulation

Consider the general linear combination:

$$
X(t) = A \cos \omega t + B \sin \omega t
$$

where $A, B \in \mathbb{R}$ are constants. The question is whether this can be expressed as a single sinusoid:

$$
X(t) = R \cos (\omega t - \phi)
$$

or equivalently

$$
X(t) = R \sin (\omega t + \delta)
$$

for some amplitude R and phase shift ϕ, δ.

Demonstration: Expressing $X(t)$ as a Single Sinusoid

Step 1: Recognize the Form

The linear combination:

$$
X(t) = A \cos \omega t + B \sin \omega t
$$

is known as a harmonic signal with coefficients A and B.

Step 2: Use the Phasor Representation

A common approach involves representing the combined sinusoid as a phasor—a complex exponential:

$$
X(t) = \Re \left\{ (A - j B) e^{j \omega t} \right\}
$$

where $j = \sqrt{-1}$.

The magnitude of the phasor is:

$$
R = \sqrt{A^2 + B^2}
$$

and the phase angle:

$$
\phi = \arctan \left(\frac{B}{A} \right)
$$

Assuming $A \neq 0$, the expression becomes:

$$
X(t) = R \cos (\omega t - \phi)
$$

\]

which is a single sinusoid with amplitude R and phase shift ϕ.

Step 3: Formal Proof

Applying the cosine difference identity:

$$
\cos (\omega t - \phi) = \cos \omega t \cos \phi + \sin \omega t \sin \phi
$$

Multiplying both sides by R:

$$
R \cos (\omega t - \phi) = R \cos \phi \cos \omega t + R \sin \phi \sin \omega t
$$

Comparing with the original expression:

$$
A \cos \omega t + B \sin \omega t
$$

we identify:

$$
A = R \cos \phi, \quad B = R \sin \phi
$$

and consequently,

$$
\boxed{
X(t) = R \cos (\omega t - \phi)
}
$$

This derivation confirms that any linear combination of $\sin \omega t$ and $\cos \omega t$ with constants can be expressed as a single sinusoid with an appropriate phase shift.

Special Case: Coefficients Equal to A

The problem states $X(t) = A \cos \omega t + A \sin \omega t$. Here, both coefficients are equal:

$$
A = B
$$

Thus,

$$
X(t) = A \cos \omega t + A \sin \omega t
$$

which simplifies to:

$$
X(t) = A (\cos \omega t + \sin \omega t)
$$

Step 1: Use the sum-to-product identities

Recall the identity:

$$
\cos \omega t + \sin \omega t = \sqrt{2} \sin \left(\omega t + \frac{\pi}{4} \right)
$$

or equivalently,

$$
\cos \omega t + \sin \omega t = \sqrt{2} \cos \left(\omega t - \frac{\pi}{4} \right)
$$

depending on the preferred phase.

Therefore,

$$
X(t) = A \sqrt{2} \sin \left(\omega t + \frac{\pi}{4} \right)
$$

or

$$
X(t) = A \sqrt{2} \cos \left(\omega t - \frac{\pi}{4} \right)
$$

This explicitly shows the linear combination reduces to a single sinusoidal function with amplitude $A \sqrt{2}$ and phase shift $\pm \frac{\pi}{4}$.

Broader Implications and Applications

Fourier Series and Signal Decomposition

The ability to combine sinusoidal functions into a single sinusoid is fundamental in Fourier analysis. It simplifies the representation of signals, allowing an infinite sum of sines and cosines to be expressed as a sum of fewer terms, often leading to more efficient analysis and processing.

System Response and Signal Modulation

In systems theory, input signals often involve linear combinations of sinusoids. Understanding that such combinations can be represented as a single sinusoid aids in analyzing system responses, especially in linear time-invariant (LTI) systems, where the output to a sinusoidal input is another sinusoid with the same frequency but different amplitude and phase.

Signal Filtering and Noise Reduction

Recognizing the phase relationships and combining sinusoidal components streamline filtering processes, allowing engineers to design filters that target specific frequency components without disturbing the overall phase relationships.

Generalization to Vectors and Complex Plane

The geometric interpretation of sinusoidal combinations can be visualized using vectors in the complex plane:

- The coefficients A and B form a vector $\vec{V} = (A, B)$.
- The magnitude $R = |\vec{V}|$ and phase ϕ specify the resultant sinusoid.
- The process of combining sinusoidal functions corresponds to vector addition, emphasizing the linearity and superposition principles.

Conclusion

This detailed investigation demonstrates that any linear association of $\sin \omega t$ and $\cos \omega t$ of the form $X(t) = A \cos \omega t + B \sin \omega t$ can be succinctly expressed as a single sinusoid with an amplitude $R = \sqrt{A^2 + B^2}$ and a phase shift $\phi = \arctan(B/A)$:

$$
X(t) = R \cos (\omega t - \phi)
$$

Specifically, for the case where $A = B$, the linear combination simplifies to a sinusoid with amplitude $A \sqrt{2}$ and phase shift $\pm \pi/4$. This property not only underscores the mathematical elegance of sinusoidal functions but also underpins practical techniques across various engineering disciplines involving waveforms, signals, and systems.

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Final Remarks

Understanding the linear association of sinusoidal functions and their reduction to a single sinusoid with phase shift is a fundamental concept with broad applications. This demonstration not only reinforces core mathematical principles but also enhances practical signal analysis techniques critical in modern engineering and physics.

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