Show That C Is Isomorphic To The Subgroup Of GL 2 (R) Consisting Of Matrices Of The Form (Ab Ba). 2.

Show That C Is Isomorphic To The Subgroup Of GL 2 (R) Consisting Of Matrices Of The Form (Ab Ba). 2.

Understanding the concept of isomorphism between groups is fundamental in algebra, particularly in the study of matrix groups and their subgroups. This article aims to demonstrate that the cyclic group C can be shown to be isomorphic to a specific subgroup of $GL_2(\mathbb{R})$—the group of all invertible 2×2 real matrices—comprising matrices of the form

$$
\begin{pmatrix}
a & b \\
b & a
\end{pmatrix}
$$

where $a, b \in \mathbb{R}$. This subgroup is particularly interesting because of its symmetric structure and its relation to rotations and reflections in the plane. The process involves constructing an explicit isomorphism, verifying its properties, and understanding the underlying algebraic structures that facilitate this correspondence.

Understanding the Cyclic Group C

Definition and Basic Properties of C

The cyclic group C is one of the simplest types of groups, generated by a single element. Formally, if g is an element of C, then

$$
C = \langle g \rangle = \{ e, g, g^2, g^3, \ldots, g^{n-1} \}
$$

where e is the identity element, and $g^n = e$ for some positive integer n. The order of C is n, and it is abelian, meaning the group operation is commutative.

For this discussion, we consider C to be a finite cyclic group of order n, commonly isomorphic to $\mathbb{Z}_n$, the integers modulo n.

The Subgroup of $GL_2(\mathbb{R})$ with Matrices of the Form $(a\,b; b\,a)$

Characterization of the Matrices

The subgroup $H \subseteq GL_2(\mathbb{R})$ consists of all invertible matrices of the form:

$$
\begin{pmatrix}
a & b \\
b & a
\end{pmatrix}
$$

with $a, b \in \mathbb{R}$. To analyze this subgroup, we first examine the conditions under which such matrices are invertible.

Invertibility Conditions

A matrix

$$
M = \begin{pmatrix}
a & b \\
b & a
\end{pmatrix}
$$

is invertible if and only if its determinant is non-zero:

$$
\det(M) = a \cdot a - b \cdot b = a^2 - b^2 \neq 0
$$

Thus, the set of matrices in H that belong to $GL_2(\mathbb{R})$ are those with $a^2 \neq b^2$.

Constructing the Isomorphism

Defining the Map $\phi: C \to H$

To establish the isomorphism, we seek a group homomorphism ϕ from C onto H that is bijective. Since C is cyclic, generated by g, the map is determined by the image of g.

Suppose $C \cong \langle g \rangle$ with $g^n = e$. We define:

$$
\phi(g^k) = M_k
$$

where M_k is a matrix of the form:

$$
M_k = \begin{pmatrix}
a_k & b_k \\
b_k & a_k
\end{pmatrix}
$$

for some real numbers a_k, b_k depending on k, such that $\phi(g^k)$ respects the group operation.

Explicit Construction of ϕ

A natural choice is to relate g to a rotation or reflection matrix. For example, consider the following:

- For g of order n, associate g with a matrix R such that $R^n = I$.

- Recall that matrices of the form

$$
\begin{pmatrix}
\cos \theta & \sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
$$

represent rotations in $\mathbb{R}^2$, with order n if $\theta = 2\pi/n$.

However, our target matrices have the form $(a\,b; b\,a)$, which can be viewed as linear combinations of the identity matrix and a symmetric matrix:

$$
\begin{pmatrix}
a & b \\
b & a
\end{pmatrix} = a \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} + b \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
$$

The matrices

$$
J = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
$$

and

$$
I = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
$$

commute, and $J^2 = I$, which suggests that the subgroup generated by J and scalar multiples of the identity could be isomorphic to a cyclic group.

Choosing the Image of g

Suppose g maps to

$$
M = a I + b J
$$

with $a, b \in \mathbb{R}$, satisfying

\[

$$
M^n = I
$$

for some n. Since $J^2 = I$, the powers of M can be computed using binomial expansion:

$$
M^k = (a I + b J)^k
$$

which simplifies due to the properties of I and J.

Verifying the Homomorphism and Isomorphism Properties

Homomorphism Property

To verify that ϕ is a homomorphism, we need to check:

$$
\phi(g^{k + l}) = \phi(g^k g^l) = \phi(g^k) \phi(g^l)
$$

which reduces to verifying:

$$
M_{k + l} = M_k M_l
$$

Given the structure of M_k, this is true if and only if the map respects the group operation, which depends on the algebraic properties of the matrices.

Injectivity and Surjectivity

- Injectivity: ϕ is injective if different elements in C map to different matrices. Since C is cyclic, and the matrices of the form $a I + b J$ with suitable a, b form a subgroup isomorphic to C, the injectivity hinges on the order of g and the minimal polynomial of the matrices.

- Surjectivity: Every matrix of the specified form with invertible determinant can be obtained from some power of g, ensuring the image covers the entire subgroup H.

Conclusion: Establishing the Isomorphism

Through the above considerations, one can explicitly construct the isomorphism:

$$
\boxed{
\phi: C \to H, \quad \text{where} \quad \phi(g^k) = a_k I + b_k J
}
$$

such that:

- a_k, b_k are chosen to satisfy $(a_k I + b_k J)^n = I$.

- The map respects group operation, i.e., $\phi(g^{k + l}) = \phi(g^k) \phi(g^l)$.

- The matrices are invertible for appropriate a_k, b_k.

This demonstrates that the cyclic group C is isomorphic to the subgroup H of $GL_2(\mathbb{R})$ consisting of matrices of the form $(a\,b; b\,a)$. The structure of these matrices, their algebraic properties, and the correspondence with the cyclic group's generator ensure the isomorphism’s validity.

Implications and Applications

Understanding this isomorphism has several applications:

- Representation Theory: It provides a concrete matrix representation of cyclic groups.

- Symmetry in Geometry: Matrices of the form $(a\,b; b\,a)$ relate to symmetries, reflections, and rotations.

- Group Classification: It illustrates how certain subgroups of $GL_2(\mathbb{R})$ are structurally equivalent to simpler

Frequently Asked Questions

What is the main idea behind showing that a group C is

isomorphic to a subgroup of GL(2, R) consisting of matrices of the form (a b; b a)?

The main idea is to construct an explicit group homomorphism from C to the subgroup of GL(2, R) with matrices of the specified form and demonstrate that this mapping is bijective, thus establishing an isomorphism.

How can we explicitly define the isomorphism between C and the subgroup of GL(2, R)?

We can define a map φ: C → GL(2, R) by assigning each element c in C to a matrix of the form (a b; b a), where the parameters a and b are functions of c, ensuring the group operation is preserved.

What properties must the matrices of the form (a b; b a) satisfy to form a subgroup of GL(2, R)?

They must be invertible matrices, which requires that the determinant $a^2 - b^2 \neq 0$, and the set must be closed under matrix multiplication and inverses, preserving the form (a b; b a).

Why is it important to verify that the map from C to the subgroup is both injective and surjective?

Because showing the map is both injective and surjective confirms it is a bijective homomorphism, which is the necessary condition for an isomorphism between the two groups.

What role does the choice of parameters a and b play in establishing the isomorphism?

The parameters a and b encode the elements of C in a way that respects the group operation, allowing us to translate the algebraic structure of C into the matrix form, thus facilitating the isomorphism.

Can you describe a specific example of an element in C and its corresponding matrix in the subgroup of GL(2, R)?

For example, if c in C corresponds to the element with parameter c, then the associated matrix could be (cosh c sinh c; sinh c cosh c), which is invertible and of the form (a b; b a), illustrating the isomorphism.

Additional Resources

Show That C Is Isomorphic To The Subgroup Of GL(2, R) Consisting Of Matrices Of The Form (a b; b a).

Introduction

In the realm of linear algebra and abstract algebra, understanding the structure of groups and their isomorphisms plays a critical role in unraveling the symmetries and transformations that underpin various mathematical and physical phenomena. One fascinating example involves demonstrating that a specific group, denoted as C, is isomorphic to a subgroup of the general linear group GL(2, R)—the group of all invertible 2×2 real matrices—comprising matrices of a particular symmetric form. This exploration not only illustrates the elegance of algebraic structures but also highlights the interconnectedness between abstract groups and concrete matrix representations.

This article aims to provide a comprehensive, detailed, and analytical exposition of this isomorphism, carefully guiding the reader through the necessary background, the construction of the subgroup, the proof of isomorphism, and the implications of this result. It is suitable for students, researchers, or enthusiasts seeking a deep understanding of algebraic structures in matrix groups.

Background Concepts and Definitions

What Is a Group?

A group is a fundamental algebraic structure consisting of a set G equipped with a binary operation (often called multiplication) satisfying four axioms:

1. Closure: For all a, b in G, the product $a \cdot b$ is also in G.
2. Associativity: For all a, b, c in G, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
3. Identity Element: There exists an element e in G such that for all a in G, $e \cdot a = a \cdot e = a$.
4. Inverse Elements: For each a in G, there exists an element a^{-1} in G such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

Examples: The set of integers under addition, the set of invertible matrices under multiplication.

What Is an Isomorphism?

An isomorphism between two groups G and H is a bijective function $\varphi: G \to H$ that respects the group operation:

$\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$ for all a, b in G.

If such a function exists, G and H are said to be isomorphic, indicating they share the same algebraic structure, even if their elements or representations differ.

The General Linear Group GL(2, R)

GL(2, R) is the group of all invertible 2×2 matrices with real entries, with matrix multiplication as the operation. Key properties:

- The identity element is the 2×2 identity matrix I.
- Inverses exist for all matrices with non-zero determinant.
- The group captures all invertible linear transformations of the 2-dimensional real vector space.

The Subgroup of Matrices of the Form (a b; b a)

We consider matrices:

$$
M(a, b) = \begin{bmatrix} a & b \\ b & a \end{bmatrix}
$$

where a and b are real numbers. To form a subgroup of GL(2, R), these matrices must be invertible:

$$
\det M(a, b) = a^2 - b^2 \neq 0.
$$

This set of matrices forms a subgroup under matrix multiplication, provided the invertibility condition is satisfied.

Understanding the Group C

Before demonstrating the isomorphism, it’s crucial to specify the group C. Typically, in such contexts, C refers to a group generated by elements that can be represented via certain algebraic relations, often related to reflections, rotations, or other symmetries. For the purpose of this discussion, suppose C is the cyclic group of order 2, i.e.,

$$
C = \{ e, c \}
$$

with the property:

$$
c^2 = e.
$$

Alternatively, if the context involves more elaborate structures, C could be a more complex group isomorphic to a subgroup of GL(2, R). For simplicity, assume C is isomorphic to the multiplicative group of two elements, which aligns naturally with the structure of matrices of the form (a b; b a).

Constructing the Correspondence: From C to the Subgroup of GL(2, R)

Step 1: Defining the Map

To demonstrate the isomorphism, we need to construct a map:

$$
\phi : C \to G
$$

where G is the subgroup of GL(2, R) consisting of matrices of the form:

$$
\begin{bmatrix} a & b \\ b & a \end{bmatrix},
$$

with $\det \neq 0$.

Suppose the elements of C are represented by two elements, say e (identity) and c, with $c^2 = e$.

Define:

$$
\phi(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,
$$

and

$$
\phi(c) = \begin{bmatrix} a & b \\ b & a \end{bmatrix}
$$

for some real numbers a, b satisfying invertibility.

Step 2: Ensuring Homomorphism Conditions

To confirm that ϕ is a group homomorphism, verify:

$$
\phi(c^2) = \phi(e) \Rightarrow \phi(c)^2 = I.
$$

Thus,

$$
\left(\begin{bmatrix} a & b \\ b & a \end{bmatrix}\right)^2 = I,
$$

which expands to:

$$
\begin{bmatrix}
a^2 + b^2 & 2ab \\
2ab & a^2 + b^2
\end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
$$

Matching entries yields the system:

$$
a^2 + b^2 = 1,
$$
$$
2ab = 0.
$$

From the second equation, either $a=0$ or $b=0$:

- If $a=0$, then from the first $b^2=1$, so $b=\pm 1$.
- If $b=0$, then $a^2=1 \Rightarrow a= \pm 1$.

Therefore, the matrices in the subgroup representing c satisfy:

$$
\text{either } a=0, b= \pm 1, \quad \text{or} \quad a= \pm 1, b=0.
$$

Step 3: Characterizing the Subgroup

The set of matrices satisfying these conditions forms a subgroup:

- When $a=0, b=\pm 1$:

$$
M_{b} = \begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix}.
$$

- When $b=0, a=\pm 1$:

$$
M_{a} = \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}.
$$

The matrices with $a=\pm 1, b=0$ form the group:

$$
\{\pm I\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\},
$$

which is isomorphic to the cyclic group of order 2.

Similarly, the matrices with $a=0, b=\pm 1$ are:

$$
\left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \right\},
$$

which also form a subgroup of order 2.

Establishing the Isomorphism

Step 1: Confirming bijectivity

The map ϕ constructed from C to the subgroup:

- Is injective: distinct elements in C map to distinct matrices, given the specific forms.
- Is surjective: every matrix of the form $(a b; b a)$ with $\det \neq 0$ corresponds to an element of C or the identity.

Step 2: Compatibility with group operation

The key is to verify that:

$$
\phi(c_1 c_2) = \phi(c_1) \phi(c_2),
$$

for all $c_1, c_2 \in C$.

Given the subgroup elements:

- $\phi(e) = I$,
- $\phi(c) = M$,

and the relations:

$$
M^2 = I,
$$

the multiplication aligns with the group law in C.

Therefore, the structure of

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