Show That The Set Of Positive Integers With Distinct Digits (in Decimal Notation) Is Finite By Finding

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Understanding the nature of numbers and their properties is a foundational aspect of number theory and combinatorics. One interesting question that often arises is: Are there finitely many positive integers that have all distinct digits in their decimal representation? This problem involves analyzing the structure of numbers, counting possibilities, and establishing upper bounds that demonstrate finiteness.

In this comprehensive article, we will explore how to prove that the set of positive integers with all distinct digits is finite by systematically finding an explicit maximum number of such integers. We will delve into the combinatorial reasoning, consider the constraints of decimal notation, and present a step-by-step approach to reach the conclusion.

Understanding the Problem: Positive Integers with Distinct Digits

Before embarking on the proof, it is essential to clarify what is meant by "positive integers with distinct digits."

Definition and Clarification

- Positive integers: All integers greater than zero (1, 2, 3, ...).
- Digits in decimal notation: The standard base-10 representation of numbers.
- Distinct digits: Each digit appears at most once in the number's decimal notation.

For example:

- 2749 has all distinct digits.
- 1023 has all distinct digits.
- 1123 does not have all distinct digits because '1' repeats.

The set in question includes all positive integers such that no digit repeats within each number.

Key Insight: The Finite Nature of the Set

At first glance, the set of such numbers might seem vast because numbers grow arbitrarily large. However, a crucial combinatorial observation limits their total quantity.

Why is the set finite?

- The digits are from the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}.
- A number with k digits cannot have more than k distinct digits.
- Since digits are from a set of 10, the number of k-digit integers with all distinct digits is finite and can be explicitly counted.
- As the number of digits increases beyond 10, it becomes impossible to have all distinct digits because there are only 10 different digits.

Thus, the set of all positive integers with distinct digits is finite because:

- For 1-digit numbers: there are 9 possibilities (1–9).
- For 2-digit numbers: there are fewer possibilities as digits cannot repeat, and the first digit cannot be zero.
- For k-digit numbers: the count diminishes as k approaches 10.
- For 11 or more digits: impossible to have all distinct digits.

Counting Positive Integers with Distinct Digits

A key step in proving finiteness is explicitly counting how many such numbers exist. This counting approach will lead to an explicit upper bound, confirming the set's finiteness.

Counting 1-Digit Numbers

- Digits: 1 through 9 (since positive integers cannot start with zero).
- Total: 9 numbers (1, 2, 3, 4, 5, 6, 7, 8, 9).

Counting 2-Digit Numbers

- First digit: 1–9 (9 options).
- Second digit: 0–9 except the first digit (9 options).
- Total: $9 \times 9 = 81$.

Counting 3-Digit Numbers

- First digit: 1–9 (9 options).
- Second digit: 0–9 excluding the first digit (9 options).
- Third digit: 0–9 excluding the first two digits (8 options).
- Total: $9 \times 9 \times 8 = 648$.

Counting 4-Digit Numbers

- First digit: 1–9 (9 options).
- Second digit: 0–9 excluding the first digit (9 options).
- Third digit: excluding the first two digits (8 options).
- Fourth digit: excluding the first three digits (7 options).
- Total: $9 \times 9 \times 8 \times 7 = 4536$.

General Counting for k-Digit Numbers

For k digits where $1 \leq k \leq 10$:

- First digit: 1–9 (9 options).
- Remaining digits: choose from remaining 9, 8, 7, ..., decreasing by 1 each time, and digits can include zero (except for the first position).

The total number of such numbers is:

$$
\text{Count}_k = 9 \times P(9, k - 1)
$$

where $P(n, r)$ is the permutation of n elements taken r at a time:

$$
P(n, r) = \frac{n!}{(n - r)!}
$$

Specifically,

\[

\text{Count}_k = 9 \times \frac{9!}{(9 - (k - 1))!}
\]

for \(1 \leq k \leq 10 \).

Summing Counts to Find the Total Number of Such Integers

The total number of positive integers with all distinct digits is:

\[
\text{Total} = \sum_{k=1}^{10} \text{Count}_k
\]

Calculating each term:

- For \(k=1 \):

\[
9 \quad (\text{digits 1–9})
\]

- For \(k=2 \):

\[
9 \times 9 = 81
\]

- For \(k=3 \):

\[
9 \times 9 \times 8 = 648
\]

- For \(k=4 \):

\[
9 \times 9 \times 8 \times 7 = 4536
\]

- For \(k=5 \):

\[
9 \times 9 \times 8 \times 7 \times 6 = 27,216
\]

- For $k=6$:

$$
9 \times 9 \times 8 \times 7 \times 6 \times 5 = 136,080
$$

- For $k=7$:

$$
9 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 = 544,320
$$

- For $k=8$:

$$
9 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 = 1,632,960
$$

- For $k=9$:

$$
9 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 = 3,265,920
$$

- For $k=10$:

$$
9 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 3,265,920
$$

Adding all these counts:

$$
\text{Total} = 9 + 81 + 648 + 4536 + 27,216 + 136,080 + 544,320 + 1,632,960 + 3,265,920 + 3,265,920
$$

Calculating sum:

$$
9 + 81 = 90
$$
$$
90 + 648 = 738
$$
$$
738 + 4,536 = 5,274
$$
\[

$$
5,274 + 27,216 = 32,490
$$
$$
32,490 + 136,080 = 168,570
$$
$$
168,570 + 544,320 = 712,890
$$
$$
712,890 + 1,632,960 = 2,345,850
$$
$$
2,345,850 + 3,265,920 = 5,611,770
$$
$$
5,611,770 + 3,265,920 = 8,877,690
$$

Therefore, the total number of positive integers with all distinct digits is exactly 8,877,690.

Conclusion: The Set Is Finite

From the counting above, it is clear that:

- The total number of positive integers with all distinct digits is finite.
- Specifically, there are no more than 8,877,690 such numbers.
- Beyond 10 digits, it is impossible to have numbers with all distinct digits because only 10 digits exist.

This explicit count provides a concrete upper bound, confirming the finiteness of the set.

Implications and Further Insights

Knowing that the set of positive integers with all distinct digits is finite has several implications:

- In combinatorics and number theory, it showcases how digit restrictions impose limitations on possible numbers.
-

Frequently Asked Questions

How can we demonstrate that the set of positive integers with distinct digits is finite?

By establishing an upper bound on the number of such integers based on the maximum number of digits, since each digit must be unique, we can show the total count is finite.

What method is used to find the total number of positive integers with all distinct digits?

The method involves counting permutations of digits for each possible length (from 1 to 10), considering that digits cannot repeat, and summing these counts to find the total number.

Why is the set of positive integers with distinct digits finite rather than infinite?

Because there's a maximum of 10 digits (0 through 9) to choose from without repetition, limiting the length and thus the total number of such integers, making the set finite.

How does factorial notation aid in counting positive integers with distinct digits?

Factorial notation helps compute permutations of digits for each length, as the number of arrangements of k digits out of n is given by n! / (n - k)! , aiding in total enumeration.

Can you outline the steps to explicitly find the total number of positive integers with all distinct digits?

Yes; first, count 1-digit numbers with distinct digits (9), then 2-digit numbers (9 × 9), up to 10-digit numbers (which is 9 × 9 × 8 × ...), and sum all these counts to obtain the total, confirming finiteness.

Additional Resources

Show That The Set Of Positive Integers With Distinct Digits (in Decimal Notation) Is Finite By Finding

Understanding the finiteness of specific sets of numbers is a fundamental aspect of number theory and combinatorics. One intriguing set to analyze is the collection of all positive integers that have distinct digits in their decimal representation. At first glance, it might seem that such a set could be infinite—after all, numbers with non-repeating digits seem plentiful. However, a careful combinatorial and logical examination reveals that this set is actually finite. The goal of this detailed review is to rigorously demonstrate this fact by systematically finding an upper bound for the size of this set.

Introduction to the Problem

The core question we aim to address is:

> Is the set of all positive integers with distinct decimal digits finite or infinite?

This set, which we denote as D, can be formally expressed as:

$$
D = \{ n \in \mathbb{N} \mid \text{digits of } n \text{ are all distinct} \}
$$

Examples include:
- Single-digit numbers: $1, 2, 3, \dots, 9$,
- Two-digit numbers with distinct digits: $12, 13, 21, 34, 98, \dots$,
- Three-digit numbers: $102, 365, 987, \dots$,
- and so forth.

The question becomes: Are there infinitely many such numbers? Or, equivalently, can we find a finite upper bound on the number of such integers?

Key Observations and Intuitive Reasoning

Before diving into formal proofs, it helps to develop some intuition:

- Digits in decimal notation: Each positive integer can be viewed as a sequence of digits $d_1, d_2, \dots, d_k$, where $d_1 \neq 0$ (since leading zeros are not written in standard notation).
- Distinct digits constraint: No digit repeats within the number.
- Maximum length: Since there are only 10 digits (0 through 9), the maximum length of a number with all distinct digits is 10 (using each digit exactly once).

This leads to an initial hypothesis: the set of positive integers with distinct digits is finite because the maximum number of digits is 10.

Formal Proof of Finiteness

Step 1: Bound on the number of digits

- Any positive integer with k digits must satisfy:
$$
10^{k-1} \leq n < 10^k
$$
- Since digits are all distinct and the total number of digits is at most 10, the maximum possible k is 10.

Conclusion:
$$
\boxed{
\text{Number of positive integers with distinct digits} \leq \sum_{k=1}^{10} \text{(Number of } k\text{-digit numbers with distinct digits)}
}
$$

Step 2: Counting k-digit numbers with distinct digits

For each k in $1 \leq k \leq 10$:

- The first digit d_1 cannot be zero; thus, it can be any digit from 1 to 9.
- The remaining $k-1$ digits are chosen from the remaining 9 digits (including zero, but excluding the digit used as the first digit), without repetition.

Counting the number of k-digit numbers with distinct digits:

$$
N_k = \text{Number of ways to select and arrange digits} = \text{(choices for first digit)} \times \text{(choices for remaining digits)}.
$$

- Choices for the first digit:

$$
9 \quad \text{(digits 1-9)}
$$

- Choices for remaining $k-1$ digits:

$$
\text{Number of permutations of } 9 \text{ remaining digits taken } (k-1) \text{ at a time} = P(9, k-1) = \frac{9!}{(9 - (k-1))!}
$$

Thus,

$$
N_k = 9 \times P(9, k-1) = 9 \times \frac{9!}{(9 - (k-1))!}
$$

For example:

- For $k=1$:

$$
N_1 = 9
$$

- For $k=2$:

$$
N_2 = 9 \times 9 = 81
$$

- For $k=3$:

$$
N_3 = 9 \times 9 \times 8 = 9 \times P(9, 2) = 9 \times 72 = 648
$$

And so forth.

Step 3: Summing over all possible lengths

Total number of positive integers with distinct digits:

$$
|D| = \sum_{k=1}^{10} N_k
$$

which is a finite sum, explicitly computable:

$$
|D| = 9 + 81 + 648 + 4536 + 27216 + 136080 + 544320 + 1,814,400 + 4,073,920 + 6,227,020
$$

(Values computed as per the permutation formulas).

Conclusion:

$$
\boxed{
\text{The set } D \text{ is finite because it contains at most }
\sum_{k=1}^{10} N_k \text{ elements.}
}
$$

Explicit Calculation of the Total Number of Such Integers

Calculating the exact total:

- $N_1 = 9$
- $N_2 = 9 \times 9 = 81$
- $N_3 = 9 \times 8 \times 7 = 504$
- $N_4 = 9 \times 8 \times 7 \times 6 = 3024$
- $N_5 = 9 \times 8 \times 7 \times 6 \times 5 = 15120$
- $N_6 = 9 \times 8 \times 7 \times 6 \times 5 \times 4 = 60480$
- $N_7 = 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 = 181440$
- $N_8 = 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 = 362880$
- $N_9 = 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 362880$
- $N_{10} = 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 0$ (but since zero can't be the first digit, and we are using all digits, the count is as above).

However, note that in the previous step, the counts for N_k were computed with the understanding that:

$$
N_k = 9 \times P(9, k-1)
$$

Therefore, the total is:

$$
|D| = \sum_{k=1}^{10} N_k
$$

Calculating explicitly:

$$
|D| = 9 + 81 + 648 + 4536 + 27216 + 136080 + 544320 + 1,814,400 + 4,073,920 + 6,227,020
$$

Adding these yields:

$$
|D| = 9 + 81 + 648 + 4536 + 27216 + 136080 + 544320 + 1,814,400 + 4,073,920 + 6,227,020 = 12,085,230
$$

Therefore, the total number of positive integers with all distinct digits is exactly 12,085,230.

Implications and Significance

The explicit enumeration confirms that:

- The set D of all positive integers with distinct digits is finite.
- The maximum possible size of this set is 12,085,230.

This is a critical conclusion with several implications:

- Finiteness of specific number sets: Sets constrained by digit properties, such as distinctness, often turn out to be finite because of inherent limitations, like the maximum number of digits.
- Combinatorial approaches: The counting strategy via permutations offers a clear, methodical way to establish bounds.
- Foundations for related problems: This finiteness result can be a stepping stone for more complex investigations, such as counting numbers with other digit restrictions or analyzing asymptotic densities.

Extensions and Related Problems

While the core problem focused on positive integers with distinct digits,

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various extensions are worth
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