Show That Tr(AB)=0 If A Is Symmetric And B Is Skew-symmetric. 5. Let AR Nn. Show That A Can Be Written

Show That Tr(AB)=0 If A Is Symmetric And B Is Skew-symmetric. 5. Let AR Nn. Show That A Can Be Written

Understanding the properties of matrices is fundamental in linear algebra, especially when dealing with symmetric and skew-symmetric matrices. This article explores a specific property involving the trace function, matrices' symmetry characteristics, and their implications. Specifically, we will demonstrate that if matrix A is symmetric and matrix B is skew-symmetric, then the trace of their product, Tr(AB), equals zero. Additionally, we will analyze matrices in the space $\mathbb{R}^n$, showing how any matrix A can be expressed as a sum of symmetric and skew-symmetric matrices.

Understanding Symmetric and Skew-symmetric Matrices

Before delving into the proof, it's essential to clarify what symmetric and skew-symmetric matrices are, along with their fundamental properties.

Symmetric Matrices

- A matrix A is symmetric if it is equal to its transpose, i.e., $A = A^T$.
- Symmetric matrices are characterized by the property $a_{ij} = a_{ji}$ for all i, j.
- They are diagonalizable with real eigenvalues and orthogonal eigenvectors.

Skew-symmetric Matrices

- A matrix B is skew-symmetric if its transpose equals its negative, i.e., $B^T = -B$.
- In terms of entries, $b_{ij} = -b_{ji}$ for all i, j.

- Diagonal entries of a skew-symmetric matrix are always zero because $b_{ii} = -b_{ii} \Rightarrow b_{ii} = 0$.

Proving that Tr(AB) = 0 When A Is Symmetric and B Is Skew-symmetric

This section provides a detailed proof that the trace of the product of a symmetric matrix A and a skew-symmetric matrix B is always zero.

Step 1: Recall the Trace Property

The trace function has several important properties:

1. Linearity: $\operatorname{Tr}(X + Y) = \operatorname{Tr}(X) + \operatorname{Tr}(Y)$
2. Similarity invariance: $\operatorname{Tr}(X Y) = \operatorname{Tr}(Y X)$
3. Trace of transpose: $\operatorname{Tr}(X^T) = \operatorname{Tr}(X)$

Note: The cyclic property $\operatorname{Tr}(XYZ) = \operatorname{Tr}(ZXY)$ is often used in proofs involving matrices.

Step 2: Express $\operatorname{Tr}(AB)$ and Use Transpose Properties

Given that $A = A^T$ (symmetric) and $B^T = -B$ (skew-symmetric), consider:

$$
\operatorname{Tr}(AB)
$$

Since the trace of a product is invariant under cyclic permutations:

$$
\operatorname{Tr}(AB) = \operatorname{Tr}(BA)
$$

Now, analyze $\operatorname{Tr}(BA)$.

Step 3: Use the Transpose to Relate $\operatorname{Tr}(AB)$ and $\operatorname{Tr}(BA)$

Note that:

$$
\operatorname{Tr}(AB) = \operatorname{Tr}[(AB)^T] = \operatorname{Tr}(B^T A^T)
$$

Since A is symmetric ($A = A^T$) and B is skew-symmetric ($B^T = -B$):

$$
\operatorname{Tr}(AB) = \operatorname{Tr}(-B A) = - \operatorname{Tr}(B A)
$$

But from earlier, $\operatorname{Tr}(AB) = \operatorname{Tr}(B A)$, so:

$$
\operatorname{Tr}(AB) = - \operatorname{Tr}(AB)
$$

which implies:

$$
2 \operatorname{Tr}(AB) = 0
$$
and thus:

$$
\boxed{
\operatorname{Tr}(AB) = 0
}
$$

Conclusion: The trace of the product of a symmetric matrix A and a skew-symmetric matrix B is zero.

Decomposition of Matrices in $\mathbb{R}^n$: Expressing Any Matrix as a Sum of Symmetric and Skew-symmetric Matrices

This section discusses a fundamental concept in linear algebra: every real square matrix can be uniquely decomposed into a symmetric and a skew-symmetric component.

Step 1: The Decomposition Theorem

Any matrix $A \in \mathbb{R}^{n \times n}$ can be expressed as:

$$
A = S + T
$$

where:

- S is symmetric ($S = S^T$)
- T is skew-symmetric ($T = - T^T$)

This decomposition is unique and can be explicitly constructed.

Step 2: Constructing S and T

Given a matrix A, define:

$$
S = \frac{A + A^T}{2}
$$

and

$$
T = \frac{A - A^T}{2}
$$

which satisfy:

- $S^T = \frac{A^T + (A^T)^T}{2} = \frac{A^T + A}{2} = S$,
- $T^T = \frac{A^T - (A^T)^T}{2} = \frac{A^T - A}{2} = -T$.

Thus, S is symmetric and T is skew-symmetric.

Step 3: Verification of the Decomposition

Verify that:

$$
A = S + T
$$

by substituting the definitions:

$$
A = \frac{A + A^T}{2} + \frac{A - A^T}{2}
$$

which simplifies to:

$$
A = \frac{2A}{2} = A
$$

confirming the correctness.

Implications of the Decomposition

- This decomposition allows for easier analysis of matrices, especially in spectral theory and optimization.
- It underpins many techniques in numerical linear algebra, such as iterative methods and matrix factorizations.
- It also provides insights into the structure of matrices and their behavior under various operations.

Applications and Significance of these Properties

Understanding these properties has several practical applications across different fields.

1. Simplification in Computations

- Recognizing that the trace of AB is zero when A is symmetric and B is skew-symmetric simplifies calculations in matrix algebra, especially in the context of quadratic forms and inner products.

2. Theoretical Insights in Physics and Engineering

- Symmetric matrices often relate to energy or stability matrices.
- Skew-symmetric matrices are associated with rotations or angular momentum.
- Their orthogonal decomposition is fundamental in mechanics and quantum physics.

3. Data Science and Machine Learning

- Symmetric matrices appear in covariance matrices, while skew-symmetric matrices relate to antisymmetric relationships.
- Decomposition techniques help in dimensionality reduction and feature extraction.

4. Optimization and Control Theory

- Matrix decompositions are used to solve optimization problems involving symmetric and antisymmetric parts of matrices.

Summary and Key Takeaways

- Symmetric and skew-symmetric matrices have distinct, well-defined properties, such as $A = A^T$ and $B^T = -B$.
- The trace of the product of a symmetric and skew-symmetric matrix is always zero: $\operatorname{Tr}(AB) = 0$.
- Any real matrix can be uniquely decomposed into symmetric and skew-symmetric parts using the formulas $S = \frac{A + A^T}{2}$ and $T = \frac{A - A^T}{2}$.
- These properties serve as fundamental tools in various mathematical and engineering disciplines, aiding in analysis, computation, and theoretical development.

Conclusion

The exploration of the relationship between symmetric and skew-symmetric matrices reveals deep structural insights into matrix algebra. Recognizing that $\operatorname{Tr}(AB) = 0$ under these conditions simplifies many problems involving matrix products. Moreover, the ability to decompose any matrix into symmetric and skew-symmetric components is a

Frequently Asked Questions

Why is the trace of the product of a symmetric matrix A and a skew-symmetric matrix B equal to zero?

Because for a symmetric matrix A and a skew-symmetric matrix B, their product AB has a trace that satisfies Tr(AB) = -Tr(AB), implying Tr(AB) = 0.

What are the key properties of symmetric and skew-symmetric matrices used to prove Tr(AB) = 0?

The key properties are that $A = A^t$ (symmetric) and $B = -B^t$ (skew-symmetric). These properties lead to the relation Tr(AB) = -Tr(AB), resulting in Tr(AB) = 0.

Can you provide a step-by-step proof that Tr(AB) = 0

when A is symmetric and B is skew-symmetric?

Yes. Starting with Tr(AB), take the transpose to get Tr(AB) = $Tr((AB)^t)$ = $Tr(B^tA^t)$. Since A is symmetric, $A^t = A$, and $B^t = -B$. Therefore, Tr(AB) = Tr((-B)A) = -Tr(BA). As trace is invariant under cyclic permutations, Tr(AB) = Tr(BA). Combining these, we get Tr(AB) = -Tr(AB), hence Tr(AB) = 0.

What does the result Tr(AB) = 0 imply about the relationship between symmetric and skew-symmetric matrices?

It indicates that symmetric and skew-symmetric matrices are orthogonal in the sense of the trace inner product, meaning their product has zero trace, reflecting a form of orthogonality in matrix space.

How can the fact that Tr(AB) = 0 be used in applications involving matrix decompositions?

This property helps in decomposing matrices into symmetric and skew-symmetric parts, and is useful in areas like spectral theory, stability analysis, and in simplifying computations involving orthogonal projections.

Given $A \in N_n$, how can A be expressed in terms of its symmetric and skew-symmetric parts?

Any matrix $A \in N_n$ can be written as the sum of its symmetric and skew-symmetric components: $A = (A + A^t)/2 + (A - A^t)/2$, where $(A + A^t)/2$ is symmetric and $(A - A^t)/2$ is skew-symmetric.

Additional Resources

Mathematical Insights into Trace Properties and Matrix Decomposition: A Deep Dive

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Introduction

Linear algebra forms the backbone of numerous scientific and engineering disciplines, offering tools to understand complex systems through matrices and their properties. Among these properties, the trace function and matrix symmetry play crucial roles in simplifying and analyzing matrix behaviors. This article delves into two fundamental topics:

1. Proving that the trace of the product of a symmetric and a skew-symmetric matrix is zero, i.e., Tr(AB) = 0 when A is symmetric and B is skew-symmetric.
2. Decomposition of a matrix A into symmetric and skew-symmetric parts, especially when A belongs to the space of all n×n real matrices, denoted as $M_n(\mathbb{R})$.

These topics are not only mathematically enriching but also foundational in understanding advanced concepts like orthogonal transformations, eigenvalue decompositions, and matrix diagonalization.

Understanding Symmetry and Skew-Symmetry in Matrices

Before delving into the proofs and decompositions, it's essential to clarify what symmetric and skew-symmetric matrices are, and how they relate to each other.

Symmetric Matrices

A real matrix A is symmetric if it satisfies:

$$
A^T = A
$$

This implies that A equals its transpose. Geometrically, symmetric matrices often represent quadratic forms and are associated with real, orthogonally diagonalizable matrices.

Skew-Symmetric Matrices

A real matrix B is skew-symmetric if:

$$
B^T = -B
$$

This property indicates that the transpose of B is its negative. Skew-symmetric matrices have zero diagonal entries because:

$$
b_{ii} = -b_{ii} \Rightarrow b_{ii} = 0
$$

which simplifies many calculations involving trace and inner products.

The Trace Function and Its Properties

The trace of an n×n matrix M, denoted Tr(M), is the sum of its diagonal entries:

$$
\operatorname{Tr}(M) = \sum_{i=1}^n m_{ii}
$$

Key properties of the trace include:

- Linearity: $\operatorname{Tr}(A + B) = \operatorname{Tr}(A) + \operatorname{Tr}(B)$
- Cyclicity: $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$, which holds for all matrices where the products are defined.
- Invariance under transpose: $\operatorname{Tr}(A) = \operatorname{Tr}(A^T)$

These properties are instrumental in proofs involving symmetry and skew-symmetry.

Proving That Tr(AB) = 0 When A Is Symmetric and B Is Skew-symmetric

The Core Theorem

Statement: If A is a symmetric matrix ($A^T = A$) and B is a skew-symmetric matrix ($B^T = -B$), then:

$$
\operatorname{Tr}(AB) = 0
$$

Step-by-Step Proof

1. Start with the trace of the product:

$$
\operatorname{Tr}(AB)
$$

2. Use the cyclic property of trace:

$$
\operatorname{Tr}(AB) = \operatorname{Tr}(BA)
$$

3. Express the transpose of the product:

$$
(AB)^T = B^T A^T
$$

Since A is symmetric and B is skew-symmetric:

$$
A^T = A,\quad B^T = -B
$$

Therefore:

$$
(AB)^T = -B A

\]

4. Observe that the trace of a matrix equals the trace of its transpose:

$$
\operatorname{Tr}(AB) = \operatorname{Tr}((AB)^T) = \operatorname{Tr}(-BA) = -\operatorname{Tr}(BA)
$$

5. But from step 2, $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$, so:

$$
\operatorname{Tr}(AB) = -\operatorname{Tr}(AB)
$$

6. Conclude that:

$$
2 \operatorname{Tr}(AB) = 0 \Rightarrow \operatorname{Tr}(AB) = 0
$$

Interpretation and Significance

This result reveals a fundamental orthogonality condition between symmetric and skew-symmetric matrices concerning the trace inner product. It underpins many aspects of matrix theory, especially in the context of Lie algebras, where the space of skew-symmetric matrices forms the Lie algebra so(n), and symmetric matrices relate to the symmetric space.

Decomposition of a Matrix in $M_n(\mathbb{R})$: Symmetric and Skew-Symmetric Parts

The Fundamental Decomposition

Any real $n \times n$ matrix A can be uniquely expressed as the sum of a symmetric matrix and a skew-symmetric matrix:

$$
A = S + T
$$

where:

- S is symmetric: $S^T = S$
- T is skew-symmetric: $T^T = -T$

Deriving the Decomposition

The decomposition is straightforward and relies on the properties of transpose:

$$
\boxed{
\begin{aligned}
S &= \frac{A + A^T}{2} \\
T &= \frac{A - A^T}{2}
\end{aligned}
}
$$

Proof:

- Symmetry of S:

$$
S^T = \left(\frac{A + A^T}{2} \right)^T = \frac{A^T + (A^T)^T}{2} = \frac{A^T + A}{2} = S
$$

- Skew-symmetry of T:

$$
T^T = \left(\frac{A - A^T}{2} \right)^T = \frac{A^T - (A^T)^T}{2} = \frac{A^T - A}{2} = - \frac{A - A^T}{2} = -T
$$

- Reconstruction of A:

$$
A = S + T
$$

This decomposition is fundamental in many areas such as spectral theory, matrix diagonalization, and numerical methods.

Uniqueness of the Decomposition

Suppose $A = S_1 + T_1 = S_2 + T_2$, with S_i symmetric and T_i skew-symmetric. Then:

$$
S_1 - S_2 = T_2 - T_1
$$

- The left is symmetric, the right is skew-symmetric.
- The only matrix that is both symmetric and skew-symmetric is the zero matrix.
- Therefore:

$$
S_1 = S_2,\quad T_1 = T_2
$$

This confirms the decomposition’s uniqueness.

Applications and Broader Context

1. Matrix Diagonalization and Spectral Theorem

Symmetric matrices are orthogonally diagonalizable, making them easier to analyze. Skew-symmetric matrices, on the other hand, have purely imaginary eigenvalues or zero, which is significant in the study of rotations and angular momentum in physics.

2. Lie Algebras and Lie Groups

The space of skew-symmetric matrices forms the Lie algebra $\mathfrak{so}(n)$, associated with the special orthogonal group $SO(n)$. The trace property that $\operatorname{Tr}(AB) = 0$ when A and B are symmetric and skew-symmetric respectively, is critical in understanding the structure and representation of these algebraic objects.

3. Quadratic Forms and Bilinear Forms

Symmetric matrices represent bilinear forms, while skew-symmetric matrices relate to alternating forms. The decomposition into symmetric and skew-symmetric parts helps in simplifying complex forms and understanding their geometric interpretations.

4. Numerical Methods

In numerical linear algebra, decomposing matrices into symmetric and skew-symmetric components allows for more stable algorithms in eigenvalue computations, matrix factorizations, and solving linear systems.

Additional Insights and Related Topics

Trace Inner Product

The trace function induces an inner product on $M_n(\mathbb{R})$:

$$
\langle A, B \rangle = \operatorname{Tr}(A^T B)
$$

This inner product respects the symmetry properties:

- Symmetric matrices form a subspace orthogonal to skew-symmetric matrices with respect to this inner product.
- The result $\operatorname{Tr}(AB) = 0$ when A is symmetric and B is skew-symmetric aligns with their orthogonality in this inner product space.

Matrix Norms and Orthogonality

The Frobenius norm, defined as:

$$
\|A\|_F = \sqrt{\operatorname{Tr}(A^T A)}
$$

is compatible with the decomposition. Since symmetric and skew-symmetric matrices are orthogonal under the

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