Solve The Initial Value Problem (2 X-6 Xy + Xy2) Dx + (1 - 3x2 + (2 + X) Y) Dy = 0, Y(1) = -4 And Then

Solve The Initial Value Problem (2 X-6 Xy + Xy2) Dx + (1 - 3x2 + (2 + X) Y) Dy = 0, Y(1) = -4 And Then

Solving initial value problems (IVPs) involving partial differential equations (PDEs) is a fundamental aspect of advanced calculus and differential equations. The given PDE,

$$
(2X - 6XY + XY^2) \, Dx + (1 - 3X^2 + (2 + X)Y) \, Dy = 0,
$$

with the initial condition $Y(1) = -4$, presents an intriguing challenge that combines methods of solving first-order PDEs with initial value specifications. In this comprehensive article, we will explore the step-by-step process to solve this PDE, analyze its structure, and interpret the solution in the context of initial conditions. This detailed explanation aims to help students, educators, and enthusiasts understand the core techniques involved, enhance their problem-solving skills, and improve their grasp of differential equations.

Understanding the Initial Value Problem (IVP)

Before diving into the solution, it is essential to understand the components of the given problem:

1. The PDE Expression

The PDE is expressed as a linear combination of differentials:

$$
P(X, Y) \, Dx + Q(X, Y) \, Dy = 0,
$$

where

$$
P(X, Y) = 2X - 6XY + XY^2,
$$

\[

$$
Q(X, Y) = 1 - 3X^2 + (2 + X)Y.
$$

2. Initial Condition

The initial condition specifies the value of Y at $X=1$:

$$
Y(1) = -4.
$$

This means that at the point $X=1$, the solution curve passes through $(1, -4)$.

Approach to Solving the PDE

The problem appears to be a first-order linear or quasi-linear PDE. To solve it, we will typically:

1. Identify the form of the PDE: Is it separable, exact, or can it be made so?
2. Check for integrability conditions: Determine if the PDE is exact or can be made exact.
3. Use characteristic equations: For first-order PDEs, the method of characteristics is often employed.
4. Apply initial conditions: Use the given initial data to find particular solutions.

Let's explore these steps in detail.

Step 1: Classify the PDE

The given PDE:

$$
(2X - 6XY + XY^2) \, Dx + (1 - 3X^2 + (2 + X)Y) \, Dy = 0,
$$

can be written as:

$$
P(X, Y) \, Dx + Q(X, Y) \, Dy = 0.
$$

\]

This is a first-order PDE, linear in differentials, but not necessarily linear in the unknown functions unless explicitly expressed as such. The structure suggests the method of characteristics may be most suitable.

Step 2: Method of Characteristics

The method of characteristics reduces the PDE to a system of ordinary differential equations (ODEs). The characteristic equations are derived from:

\[
\frac{dY}{dX} = \frac{Q(X, Y)}{P(X, Y)}.
\]

Given the PDE, the characteristic ODE is:

\[
\frac{dY}{dX} = \frac{1 - 3X^2 + (2 + X)Y}{2X - 6XY + XY^2}.
\]

Our goal now is to simplify this ratio to a form suitable for solving.

Step 3: Simplify the Characteristic Equation

Let's analyze numerator and denominator separately.

Numerator:

\[
N(X, Y) = 1 - 3X^2 + (2 + X)Y,
\]

Denominator:

\[
D(X, Y) = 2X - 6XY + XY^2.
\]

Note that the denominator has a common factor of \(X\):

\[

$$
D(X, Y) = X(2 - 6Y + Y^2).
$$

Similarly, the numerator can be viewed as a linear function in Y:

$$
N(X, Y) = 1 - 3X^2 + (2 + X)Y.
$$

The ratio becomes:

$$
\frac{dY}{dX} = \frac{1 - 3X^2 + (2 + X)Y}{X(2 - 6Y + Y^2)}.
$$

Step 4: Variable Substitution to Simplify the Equation

To solve this nonlinear first-order ODE, a substitution can help. Notice the quadratic in Y in denominator: $2 - 6Y + Y^2$. Let's analyze this quadratic:

$$
Y^2 - 6Y + 2.
$$

Complete the square:

$$
Y^2 - 6Y + 9 - 7 = (Y - 3)^2 - 7.
$$

Thus:

$$
2 - 6Y + Y^2 = (Y - 3)^2 - 7 + 2 = (Y - 3)^2 - 5.
$$

The characteristic equation becomes:

$$
\frac{dY}{dX} = \frac{1 - 3X^2 + (2 + X)Y}{X[(Y - 3)^2 - 5]}.
$$

This hints that the substitution $Z = Y - 3$ could simplify the quadratic expression.

Step 5: Substituting $Z = Y - 3$

Let:

$$
Z = Y - 3,
$$
then:

$$
Y = Z + 3,
$$
and the numerator:

$$
N(X, Z) = 1 - 3X^2 + (2 + X)(Z + 3).
$$

Expanding:

$$
N(X, Z) = 1 - 3X^2 + (2 + X)Z + 3(2 + X) = 1 - 3X^2 + (2 + X)Z + 6 + 3X.
$$

Combine constants:

$$
1 + 6 + (-3X^2) + 3X + (2 + X)Z = 7 - 3X^2 + 3X + (2 + X)Z.
$$

The denominator:

$$
D(X, Z) = X[(Y - 3)^2 - 5] = X[Z^2 - 5].
$$

Now, the derivative:

$$
\frac{dY}{dX} = \frac{N(X, Z)}{D(X, Z)} = \frac{7 - 3X^2 + 3X + (2 + X)Z}{X(Z^2 - 5)}.
$$

But since $Y = Z + 3$, then:

$$
\frac{dY}{dX} = \frac{dZ}{dX},

\]

leading to the differential equation:

\[
\frac{dZ}{dX} = \frac{7 - 3X^2 + 3X + (2 + X)Z}{X(Z^2 - 5)}.
\]

This is a first-order nonlinear ODE in \(Z\) and \(X\). Next, we analyze the form to find a solution.

Step 6: Recognize the Structure and Find a Suitable Method

The differential equation:

\[
\frac{dZ}{dX} = \frac{A(X) + B(X) Z}{X(Z^2 - 5)},
\]

where

\[
A(X) = 7 - 3X^2 + 3X, \quad B(X) = 2 + X.
\]

This resembles a Bernoulli or Riccati type, but with a quadratic in \(Z\) in the denominator. To proceed, consider the substitution:

\[
W = Z^2,
\]
so that:

\[
\frac{dW}{dX} = 2 Z \frac{dZ}{dX}.
\]

Expressed in terms of \(W\):

\[
\frac{dW}{dX} = 2 Z \cdot \frac{A(X) + B(X) Z}{X(Z^2 - 5)} = \frac{2 Z [A(X) + B(X) Z]}{X(Z^2 - 5)}.
\]

But \(W = Z^2\), so:

$$
Z = \pm \sqrt{W}.
$$

Substituting back:

$$
\frac{dW}{dX} = \frac{2 \sqrt{W} [A(X) + B(X) \sqrt{W}]}{X(W - 5)}.
$$

This still appears complex, indicating that perhaps an alternative substitution or integrating factor approach is necessary.

Step 7

Frequently Asked Questions

What type of differential equation is $(2x - 6xy + xy^2) Dx + (1 - 3x^2 + (2 + x)y) Dy = 0$?

It is a first-order, nonlinear, differential equation that can potentially be classified as an exact differential equation or solved using substitution methods.

How can I check if the differential equation is exact?

By comparing the partial derivatives of the functions multiplying Dx and Dy: if $\partial M/\partial y = \partial N/\partial x$, where M and N are the coefficients of Dx

and Dy respectively, then the equation is exact.

What is the initial condition provided, and how does it help in solving the differential equation?

The initial condition is Y(1) = -4. It helps determine the particular solution after integrating the general solution of the differential equation.

What steps are involved in solving this initial value problem?

First, verify if the equation is exact or find an integrating factor; second, integrate to find the general solution; third, apply the initial condition to determine the specific solution.

Can substitution simplify solving this differential equation?

Yes, substitution methods such as setting u = xy or other strategic substitutions may reduce the equation to a separable or linear form, simplifying the solution process.

What is an integrating factor, and how is it used here?

An integrating factor is a function that, when multiplied through the differential equation, makes it exact. It's used when the equation isn't initially exact but can be converted into an exact equation.

Once the general solution is found, how do I apply the initial condition Y(1) = -4?

Substitute x = 1 and y = -4 into the general solution to solve for any arbitrary constants, thus obtaining the particular solution that satisfies the initial condition.

Are there common pitfalls to avoid when solving such differential equations?

Yes, common pitfalls include neglecting to verify if the equation is exact, incorrectly computing partial derivatives, and misapplying initial conditions. Careful algebraic manipulation and verification are essential.

What tools or software can assist in solving complex

initial value problems like this?

Software such as Wolfram Mathematica, Maple, or online tools like Wolfram Alpha can help perform symbolic calculations, verify solutions, and visualize the behavior of solutions.

Additional Resources

Solve The Initial Value Problem (2X - 6XY + XY^2) Dx + (1 - $3X^2$ + (2 + X)Y) Dy = 0, Y(1) = -4 And Then

---

Introduction

In the realm of differential equations, solving initial value problems (IVPs) stands as a cornerstone activity that bridges theoretical mathematics with practical applications across physics, engineering, and other sciences. The specific IVP under consideration:

\[
(2X - 6XY + XY^2) \, dX + (1 - 3X^2 + (2 + X)Y)

\, dY = 0, \quad \text{with} \quad Y(1) = -4,
\]

presents an intriguing challenge due to its non-linear and coupled nature. This problem exemplifies the complexities encountered when addressing real-world systems where variables are intertwined, and solutions demand nuanced approaches.

This article offers a comprehensive exploration of methods to solve this IVP, including an analytical breakdown of the differential equation, identification of potential solution strategies, step-by-step solution derivation, and an analytical discussion of the solution's properties. The goal is to provide clarity not only on the specific problem but also on the broader techniques employed in similar differential equation scenarios.

Understanding the Differential Equation

General Form and Components

The differential equation is expressed as:

\[

$$
(2X - 6XY + XY^{2}) \, dX + (1 - 3X^{2} + (2 + X)Y) \, dY = 0.
$$

This can be viewed as a first-order, non-linear differential equation with variables X and Y. The structure suggests that it could be either exact or transformable into an exact form, or potentially reducible via substitution strategies.

Rearranged, the equation takes the form:

$$
M(X, Y) \, dX + N(X, Y) \, dY = 0,
$$

where

$$
M(X, Y) = 2X - 6XY + XY^{2},
$$

$$
N(X, Y) = 1 - 3X^{2} + (2 + X)Y.
$$

The initial condition $Y(1) = -4$ indicates that at $X=1$, Y takes the value -4. Our goal is to find a function $Y(X)$ satisfying the differential equation and this initial condition.

Step 1: Analyzing the Differential Equation's Nature

Exactness and Integrating Factors

A common first step in solving differential equations of the form $M \, dX + N \, dY = 0$ is to check whether the equation is exact, i.e., whether

$$
\frac{\partial M}{\partial Y} = \frac{\partial N}{\partial X}.
$$

Calculating these derivatives:

$$
\frac{\partial M}{\partial Y} = -6X + 2XY,
$$
$$
\frac{\partial N}{\partial X} = -6X + Y.
$$

Comparing:

$$
-6X + 2XY \quad \text{vs.} \quad -6X + Y,
$$

they are not equal unless specific conditions hold, which they do not in general. Therefore, the equation is not exact as-is.

Search for an Integrating Factor

Given the non-exactness, the next step is to consider possible integrating factors—functions multiplying the entire equation to make it exact. These are often functions of X, Y, or combinations thereof.

- Possible integrating factors:

- Function of X alone
- Function of Y alone
- Function of XY or other combinations

Investigation of such potential factors involves checking whether multiplying M and N by a function $\mu(X, Y)$ can satisfy the exactness condition.

Step 2: Simplifying the Equation via

Substitution

Given the structure of \(M\) and \(N\), substitution strategies seem promising.

Observations about \(M\) and \(N\):

- The term \(XY^2\) in \(M\) and \((2 + X)Y\) in \(N\) suggest that factors involving \(XY\) or \(Y\) could simplify the form.
- The presence of quadratic terms in \(X\) and \(Y\) indicates that substitution could linearize or reduce the order of nonlinearity.

Candidate substitution: Let \(Z = XY\)

Define:

\[
Z = XY,
\]
which implies:

\[
Y = \frac{Z}{X}.
\]

Differentiating \(Y\) with respect to \(X\):

\[

\frac{dY}{dX} = \frac{d}{dX} \left(\frac{Z}{X} \right) = \frac{X \frac{dZ}{dX} - Z}{X^2} = \frac{\frac{dZ}{dX} - \frac{Z}{X}}{X}.
\]

Expressing \(dY\):

\[
dY = \frac{\frac{dZ}{dX} - \frac{Z}{X}}{X} dX.
\]

Now, substitute back into the original differential equation.

Step 3: Rewriting the Differential Equation in Terms of \(Z\)

Express \(M\) and \(N\) in terms of \(Z\) and \(X\):

\[
M = 2X - 6XY + XY^2 = 2X - 6Z + Z \frac{Z}{X} = 2X - 6Z + \frac{Z^2}{X}.
\]

Similarly,

\[

$$
N = 1 - 3X^2 + (2 + X)Y = 1 - 3X^2 + (2 + X) \frac{Z}{X} = 1 - 3X^2 + \left(\frac{2 + X}{X} \right) Z.
$$

The differential form becomes:

$$
\left(2X - 6Z + \frac{Z^2}{X} \right) dX + \left[1 - 3X^2 + \left(\frac{2 + X}{X} \right) Z \right] dY = 0.
$$

Substituting dY:

$$
dY = \frac{\frac{dZ}{dX} - \frac{Z}{X}}{X} dX.
$$

Thus, the differential equation in terms of Z and X becomes:

$$
\left(2X - 6Z + \frac{Z^2}{X} \right) dX + \left[1 - 3X^2 + \left(\frac{2 + X}{X} \right) Z \right] \frac{\frac{dZ}{dX} - \frac{Z}{X}}{X} dX = 0.
$$

Factor dX out:

$$
\left[2X - 6Z + \frac{Z^2}{X} + \left(1 - 3X^2 + \left(\frac{2 + X}{X} \right) Z \right) \frac{\frac{dZ}{dX} - \frac{Z}{X}}{X} \right] dX = 0.
$$

Dividing through by dX:

$$
2X - 6Z + \frac{Z^2}{X} + \left(1 - 3X^2 + \left(\frac{2 + X}{X} \right) Z \right) \frac{\frac{dZ}{dX} - \frac{Z}{X}}{X} = 0.
$$

This expression is quite complex, but it suggests that the substitution translates the original non-linear problem into an equation in Z and X that might be more manageable.

Step 4: Alternative Approach – Recognizing a Bernoulli Equation

Given the complexity, it's prudent to analyze whether the original differential equation reduces to a Bernoulli form or other recognizable types.

Expressing the original equation in Y and X:

Divide the entire differential equation by dX:

$$
M(X, Y) + N(X, Y) \frac{dY}{dX} = 0,
$$

which rearranged gives:

$$
\frac{dY}{dX} = - \frac{M(X, Y)}{N(X, Y)}.
$$

Plugging in the expressions:

$$
\frac{dY}{dX} = - \frac{2X - 6XY + XY^2}{1 - 3X^2 + (2 + X)Y}.
$$

Factor numerator:

$$
2X - 6XY + XY^2 = 2X - 6XY + XY^2,
$$

which can be written as:

$$
XY^2 - 6XY + 2X,
$$
or in terms of Y, as a quadratic

[Solve The Initial Value Problem 2 X 6 Xy Xy2 Dx 1 3x2 2 X Y Dy 0 Y 1 4 And Then]

Solve The Initial Value Problem 2 X 6 Xy Xy2 Dx 1 3x2 2 X Y Dy 0 Y 1 4 And Then

Related Articles

- [The Students At Walden Middle School Are Selling Tins Of Popcorn To Raise Money For New Uniforms. They]
- [The Author Also Uses __________ To Describe The Building And To Give The Reader A Sense Of "forbidding"]
- [This Is The Graph Of F(x) = X+A/x+B. The Grid Lines In The Graph Are One Unit Apart. What Is The Value]

[Back to Home]