Solve The Initial Value Problem. Give The Explicit Solution \(Y=f(x) \) \[\left(y^{3}-1\right) E^{x}

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Solving initial value problems (IVPs) is a fundamental aspect of differential equations, essential in fields ranging from physics and engineering to economics and biology. An initial value problem involves finding a function \(y(x) \) that satisfies a differential equation and also meets a specific initial condition. In this comprehensive guide, we will explore the process of solving an IVP to derive an explicit solution \(Y=f(x) \) based on the differential equation involving \(\left(y^{3}-1\right) E^{x} \). We will break down the steps, describe key concepts, and provide practical examples to enhance understanding.

Understanding the Initial Value Problem (IVP)

What is an Initial Value Problem?

An initial value problem is a differential equation paired with an initial condition specifying the value of the unknown function at a particular point. Formally, an IVP typically looks like:

\[
\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0
\]

where:
- \(\frac{dy}{dx} \) is the derivative of \(y \) with respect to \(x \),
- \(f(x, y) \) is a given function,
- \(x_0 \) is the initial point,
- \(y_0 \) is the initial value of \(y \) at \(x = x_0 \).

The goal is to find a function \(y = y(x) \) that satisfies both the differential equation and the initial condition.

Importance of Solving IVPs

Solving IVPs is crucial because:
- They model real-world phenomena with specific initial conditions.
- They enable predictions and simulations in science and engineering.
- Explicit solutions provide direct formulas for \(y(x) \), facilitating analysis and computation.

Examining the Differential Equation $(y^3 - 1) E^{x}$

Before solving, understanding the structure of the given differential equation is essential.

Form of the Differential Equation

The differential equation in question involves the expression:

$$
\frac{dy}{dx} = (y^3 - 1) E^{x}
$$

This is a first-order, nonlinear differential equation because of the y^3 term.

Key Features

- The right-hand side is a product of a function of y ($y^3 - 1$) and an exponential function of x (E^{x}).
- It resembles a separable differential equation, which can be rearranged to isolate variables.

Step-by-Step Solution of the Initial Value Problem

The process involves several steps:
1. Rearranging the equation to separate variables.
2. Integrating both sides.
3. Applying the initial condition to find the constant of integration.
4. Expressing the explicit solution $y=f(x)$.

Let's delve into each step with detailed explanations.

Step 1: Separating Variables

Starting with:

$$
\frac{dy}{dx} = (y^3 - 1) E^{x}
$$

we aim to write all y-terms on one side and all x-terms on the other:

$$
\frac{dy}{y^3 - 1} = E^{x} dx
$$

This step assumes that $y^3 - 1 \neq 0$, which is valid except at the specific point $y=1$. The separation of variables is valid for all other y-values.

Step 2: Integrate Both Sides

Now, integrate both sides:

$$
\int \frac{dy}{y^3 - 1} = \int E^{x} dx
$$

The right side integrates straightforwardly:

$$
\int E^{x} dx = E^{x} + C
$$

where C is the constant of integration.

The left side requires partial fraction decomposition because $y^3 - 1$ factors as:

$$
y^3 - 1 = (y - 1)(y^2 + y + 1)
$$

Thus,

$$
\frac{1}{y^3 - 1} = \frac{A}{y - 1} + \frac{By + C}{y^2 + y + 1}
$$

Step 3: Partial Fraction Decomposition

Find constants A, B, C such that:

\[

$$
\frac{1}{(y - 1)(y^2 + y + 1)} = \frac{A}{y - 1} + \frac{By + C}{y^2 + y + 1}
$$

Multiply both sides by the denominator:

$$
1 = A(y^2 + y + 1) + (By + C)(y - 1)
$$

Expand:

$$
1 = A y^2 + A y + A + B y (y - 1) + C (y - 1)
$$

$$
= A y^2 + A y + A + B y^2 - B y + C y - C
$$

Group like terms:

$$
(A y^2 + B y^2) + (A y - B y + C y) + (A - C) = 1
$$

Simplify:

$$
(A + B) y^2 + (A - B + C) y + (A - C) = 1
$$

Since the right side is constant, the coefficients of y^2 and y must be zero:

$$
A + B = 0 \implies B = -A
$$

$$
A - B + C = 0
$$

And the constant term:

$$
A - C = 1

\]

Substitute \(B = -A \) into the second equation:

\[
A - (-A) + C = 0 \implies A + A + C = 0 \implies 2A + C = 0 \implies C = -2A
\]

From the third equation:

\[
A - C = 1
\]

Substitute \(C = -2A \):

\[
A - (-2A) = 1 \implies A + 2A = 1 \implies 3A = 1 \implies A = \frac{1}{3}
\]

Then,

\[
B = -A = -\frac{1}{3}
\]
\[
C = -2A = -\frac{2}{3}
\]

Now, rewrite the integral:

\[
\int \frac{dy}{y^3 - 1} = \int \left(\frac{1/3}{y - 1} + \frac{(-1/3) y - 2/3}{y^2 + y + 1} \right) dy
\]

Split into separate integrals:

\[
\int \frac{1/3}{y - 1} dy + \int \frac{(-1/3) y - 2/3}{y^2 + y + 1} dy
\]

Pull out constants:

$$
\frac{1}{3} \int \frac{1}{y - 1} dy - \frac{1}{3} \int \frac{ y + 2 }{ y^2 + y + 1 } dy
$$

Note: The numerator in the second integral is $- (y + 2)$ because of the negative sign; carefully tracking signs:

Actually, the second integral is:

$$
\int \frac{ - \frac{1}{3} y - \frac{2}{3} }{ y^2 + y + 1 } dy
$$

which can be written as:

$$
- \frac{1}{3} \int \frac{ y + 2 }{ y^2 + y + 1 } dy
$$

since the numerator is $- (y + 2)$.

Step 4: Final Integration

Now, the integral becomes:

$$
\frac{1}{3} \int \frac{1}{y - 1} dy - \frac{1}{3} \int \frac{ y + 2 }{ y^2 + y + 1 } dy = E^{x} + C
$$

Integral 1:

$$
\int \frac{1}{y - 1} dy = \ln | y - 1 |
$$

Integral 2:

For:

\[

\int \frac{ y + 2 }{ y^2 + y + 1 } dy
\]

we perform substitution:

- Observe numerator \(y + 2 \), which can be related to the derivative of the denominator:

\[
\frac{d}{dy} (y^2 + y + 1) = 2 y + 1
\]

Express numerator \(

Frequently Asked Questions

What is the initial value problem given in the equation?

The initial value problem is to solve the differential equation $(y^3 - 1) e^x$ and find the explicit solution $y = f(x)$.

How do you approach solving the differential equation $(y^3 - 1) e^x = dy/dx$?

Rearrange the equation to isolate dy/dx, then separate variables if possible, and integrate both sides to find y as a function of x.

What substitution can simplify solving the differential equation $(y^3 - 1) e^x = dy/dx$?

A common substitution is $u = y^3 - 1$, which transforms the equation into a separable form, making it easier to integrate.

What is the explicit solution $y = f(x)$ for the differential equation?

The explicit solution is $y = [1 + C e^{x}]^{1/3}$, where C is the constant of integration determined by initial conditions.

How do initial conditions affect the solution of this differential equation?

Initial conditions allow you to solve for the constant C in the explicit solution, providing a specific solution $y = f(x)$.

Is the solution $y = [1 + C e^{x}]^{1/3}$ valid for all x?

The solution is valid where the expression inside the cube root is defined and real; specifically, $1 + C e^{x} > 0$ for real solutions.

What are common applications of solving differential equations like this one?

Such equations appear in modeling growth processes, chemical reactions, and physical systems where rate changes depend on the current state.

What are potential challenges when solving this type of differential equation?

Challenges include managing the algebraic manipulation, ensuring the solution remains within the domain where it is defined, and applying initial conditions correctly.

Additional Resources

Solve The Initial Value Problem: Unveiling the Explicit Solution $Y = f(x)$ for $(y^3 - 1) e^x$

Introduction

When approaching differential equations, especially initial value problems (IVPs), mathematicians and engineers alike seek explicit solutions that allow for straightforward analysis and application. These solutions serve as the backbone for modeling real-world phenomena—from physics and engineering to economics and biology. Today, we delve into a specific initial value problem characterized by the differential expression:

$$
(y^3 - 1) e^x
$$

Our goal is to find the explicit solution $Y = f(x)$ that satisfies the initial condition, uncovering the methodical steps, underlying principles, and implications of this solution. Think of this as a meticulously crafted product review—each component examined to understand its value, functionality, and intricacies.

Understanding the Problem

The Differential Equation

The problem involves a differential equation that, based on the provided expression, likely takes the form:

$$
\frac{dy}{dx} = (y^3 - 1) e^x
$$

This is a first-order ordinary differential equation (ODE), with a nonlinear term y^3. The presence of e^x suggests an exponential growth factor, while the cubic term in y indicates nonlinearity, which adds complexity but also richness to the solution process.

Initial Condition

An initial condition, say $y(x_0) = y_0$, is essential to specify a unique solution. While the problem statement does not explicitly provide this, solving the IVP typically proceeds with such a condition, for example:

$$
y(0) = y_0
$$

The choice of y_0 influences the particular solution, guiding the integration constant that emerges during the solution process.

Step-by-Step Solution Approach

Approaching this IVP involves systematic methods—most notably, separable differential equations and integration.

1. Recognizing the Equation Type

The differential equation:

$$
\frac{dy}{dx} = (y^3 - 1) e^x
$$

is separable, meaning it can be expressed in the form:

\[
\frac{dy}{f(y)} = g(x) dx
\]

which allows us to integrate both sides independently.

2. Rearranging for Separation of Variables

Rearranged, the equation becomes:

\[
\frac{dy}{y^3 - 1} = e^x dx
\]

This setup isolates the variables, setting the stage for integration.

Integration Process

3. Decomposing the Left Side: Partial Fraction Decomposition

The integral:

\[
\int \frac{dy}{y^3 - 1}
\]

requires factoring the denominator:

\[
y^3 - 1 = (y - 1)(y^2 + y + 1)
\]

The quadratic \(y^2 + y + 1 \) does not factor further over the reals, so partial fractions are applied:

\[
\frac{1}{(y - 1)(y^2 + y + 1)} = \frac{A}{y - 1} + \frac{By + C}{y^2 + y + 1}
\]

Multiplying both sides by the denominator:

\[

1 = A(y^2 + y + 1) + (By + C)(y - 1)
\]

Expanding:

\[
1 = A y^2 + A y + A + B y^2 - B y + C y - C
\]

Grouping like terms:

\[
1 = (A + B) y^2 + (A - B + C) y + (A - C)
\]

Matching coefficients with the constant term 1:

\[
\begin{cases}
A + B = 0 \\
A - B + C = 0 \\
A - C = 1
\end{cases}
\]

Solving this system:

- From \(A + B = 0 \), \(B = -A \).
- From \(A - B + C = 0 \), substitute \(B \):

\[
A - (-A) + C = 0 \Rightarrow A + A + C = 0 \Rightarrow 2A + C = 0
\]

- From \(A - C = 1 \):

\[
A - C = 1 \Rightarrow C = A - 1
\]

Substitute into \(2A + C = 0 \):

\[

2A + (A - 1) = 0 \Rightarrow 3A - 1 = 0 \Rightarrow A = \frac{1}{3}
\]

Then:

\[
C = \frac{1}{3} - 1 = -\frac{2}{3}
\]

And:

\[
B = -A = -\frac{1}{3}
\]

4. Expressing the Partial Fractions

Putting it all together:

\[
\frac{1}{(y - 1)(y^2 + y + 1)} = \frac{\frac{1}{3}}{y - 1} + \frac{-\frac{1}{3} y - \frac{2}{3}}{y^2 + y + 1}
\]

Simplify numerator:

\[
\frac{1}{3} \cdot \frac{1}{y - 1} - \frac{\frac{1}{3} y + \frac{2}{3}}{y^2 + y + 1}
\]

Integrating Both Sides

5. Integrate the Left Side

\[
\int \left(\frac{1/3}{y - 1} - \frac{\frac{1}{3} y + \frac{2}{3}}{y^2 + y + 1} \right) dy
\]

Splitting the integral:

\[
\frac{1}{3} \int \frac{dy}{y - 1} - \int \frac{\frac{1}{3} y + \frac{2}{3}}{y^2 + y + 1} dy

\]

The first integral:

\[
\frac{1}{3} \ln | y - 1 |
\]

The second integral involves completing the square in the denominator:

\[
y^2 + y + 1 = \left(y + \frac{1}{2} \right)^2 + \frac{3}{4}
\]

Express numerator in terms of \(y + \frac{1}{2} \):

\[
\frac{1}{3} y + \frac{2}{3} = \frac{1}{3} \left(y + 2 \right)
\]

But note, better to write \(y \) in terms of \(y + \frac{1}{2} \):

\[
y = \left(y + \frac{1}{2} \right) - \frac{1}{2}
\]

Thus:

\[
\frac{1}{3} y + \frac{2}{3} = \frac{1}{3} \left(y + \frac{1}{2} \right) - \frac{1}{6} + \frac{2}{3}
\]

Simplify:

\[
\frac{1}{3} \left(y + \frac{1}{2} \right) + \left(-\frac{1}{6} + \frac{2}{3} \right) = \frac{1}{3} \left(y + \frac{1}{2} \right) + \frac{1}{2}
\]

Now, the integral becomes:

\[
- \int \frac{\frac{1}{3} \left(y + \frac{1}{2} \right) + \frac{1}{2}}{\left(y + \frac{1}{2} \right)^2 +

\frac{3}{4}} dy
\]

Let \(u = y + \frac{1}{2} \), then \(du = dy \).

Change of variables:

\[
- \int \frac{\frac{1}{3} u + \frac{1}{2}}{u^2 + \frac{3}{4}} du
\]

which separates into:

\[
- \frac{1}{3} \int \frac{u}{u^2 + \frac{3}{4}} du - \frac{1}{2} \int \frac{1}{u^2 + \frac{3}{4}} du
\]

Final Integration and Explicit Solution

6. Computing the Integrals

- For \(\int \frac{u}{u^2 + a^2} du \):

\[
\frac{1}{2} \ln (u^2 + a^2)
\]

- For \(\int \frac{1}{u^2 + a^2} du \):

\[
\frac{1}{a} \arctan \left(\frac{u}{a} \right)
\]

where \(a^2 = \frac{3}{4} \Rightarrow a = \frac{\sqrt{3}}{2}

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