Suppose AB=AC, Where And C Are Nxp Matrices And Is Invertible. Show That B=C_ Is This True In General,

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Understanding the properties of matrices is fundamental in linear algebra, especially when analyzing relationships involving invertible matrices and their implications for other matrices. In particular, the question of whether AB = AC implies B = C hinges on the invertibility of certain matrices involved. This piece explores the scenario where matrices A, B, and C are all n×p matrices, with C being invertible, and examines whether the equality AB = AC necessarily leads to B = C. We will analyze this question comprehensively, considering mathematical proofs, counterexamples, and the conditions under which this statement holds or fails.

Background and Context

Before delving into the proof or disproof of the statement, it is essential to establish the context and recall some fundamental concepts.

Matrix Dimensions and Notation

- A, B, C: Matrices of dimensions n×p.
- Invertibility: A matrix C is invertible (or nonsingular) if there exists a matrix C^{-1} such that C C^{-1} = C^{-1} C = I, where I is the identity matrix of appropriate size.

Key Assumptions

- Matrices A, B, C are all n×p matrices.
- C is invertible; thus, C^{-1} exists.
- The equation AB = AC holds for the matrices involved.

The Core Question

The central question is:

> Given that AB = AC and that C is invertible, does it necessarily follow that B = C? Or, more generally, does AB = AC imply B = C_?

Note that the notation C_ is ambiguous in the initial statement. It seems to suggest that B equals some transformation involving C, perhaps $B = C^{-1} B$ or similar, but in typical matrix algebra, the most natural interpretation is whether B equals $C^{-1} C$, which simplifies to B = C.

Therefore, the question reduces to:

> If AB = AC and C is invertible, does that imply B = C?

This interpretation aligns with common linear algebra problems about the cancelation property involving invertible matrices.

Analysis of the Statement: AB = AC with C Invertible

Step 1: Express B in terms of A and C

Given AB = AC and C is invertible, we can attempt to isolate B:

$$
AB = AC
$$

Multiplying both sides on the right by C^{-1}:

$$
A B C^{-1} = A C C^{-1} = A I = A
$$

But this step alone does not directly give B, because B is multiplied on the right by C, not necessarily on the left. To proceed, we need to consider the properties of A.

Step 2: Conditions on A for cancelation

- If A is invertible, then:

$$
A B = A C \implies B = C
$$

since multiplying both sides on the left by A^{-1} yields:

$$
A^{-1} A B = A^{-1} A C \implies B = C
$$

- However, if A is not invertible, this cancelation step is invalid, and B may not equal C even if AB = AC.

Summary of the key point:

- The invertibility of A is crucial for concluding B = C from AB = AC.
- When A is invertible:

$$
AB = AC \implies B = C
$$

- When A is not invertible:

$$
AB = AC \not\Rightarrow B = C \quad \text{necessarily}
$$

Therefore, the statement "Suppose AB=AC, where C are invertible, then B=C" is true if A is also invertible, but not necessarily otherwise.

Counterexamples and Limitations

To understand whether the statement holds in general, consider scenarios where A is not invertible.

Counterexample 1: Non-invertible A

Let’s choose specific matrices:

$$
A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (\text{zero matrix}),
$$
$$
B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},
$$
$$
C = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}.
$$

Calculate:

$$
AB = \text{zero matrix} \implies AB = 0,
$$
$$
AC = 0,
$$

so

$$
AB = AC \implies 0 = 0,
$$

which is always true, regardless of B and C. But here, $B \neq C$, indicating that the initial matrices B and C can be different even when $AB=AC$ with $A=0$. Thus, the equality does not guarantee $B = C$ in this case.

Counterexample 2: A is singular but not zero

Suppose:

$$
A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},
$$
$$
B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix},
$$
$$
C = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}.
$$

Calculate:

$$
AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix},
$$
$$
AC = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix},
$$

which are equal, yet $B \neq C$. This demonstrates that when A is not invertible, the equality $AB=AC$ does not imply $B=C$.

Conclusion from counterexamples:
The invertibility of A is necessary for the cancelation property.

Summary of Conditions for B = C

Condition	Implication	Explanation
A is invertible	Yes	$AB=AC \Rightarrow B=C$
C is invertible	Not sufficient	C invertibility alone does not imply $B=C$ without A being invertible

A is not invertible	No	Counterexamples show $B \neq C$ can still satisfy $AB=AC$

Hence, the initial statement "Suppose AB=AC, where C are invertible matrices, show that B=C" is true if and only if matrix A is also invertible.

Generalization and Additional Considerations

Beyond the specific case of matrices over real or complex fields, similar properties hold in more general algebraic structures, but the core principle remains:

- Cancelation Law: In a ring or algebra where invertibility holds, the cancelation law applies only when the element being canceled (here, A) is invertible.

Implication for Linear Algebra Practice

- When solving matrix equations, always verify the invertibility of the matrices involved before canceling terms.
- Be cautious in assuming equality of matrices from the equality of their products unless invertibility justifies it.

Conclusion and Final Remarks

In conclusion, the statement "Suppose AB=AC, where C are invertible matrices, show that B=C" is conditionally true in general. The critical condition is the invertibility of matrix A:

- If A is invertible, then from AB=AC, multiplying both sides on the left by A^{-1} yields:

$$
A^{-1} AB = A^{-1} AC \implies B = C
$$

- If A is not invertible, the equality AB=AC does not necessarily imply B=C, as shown through counterexamples.

Therefore, the general statement holds true only under the assumption that A is invertible.

This analysis underscores the importance of matrix invertibility in linear algebraic equations and the implications for matrix equality and cancelation properties.

Meta Note: This comprehensive discussion provides a detailed understanding of the conditions under which matrix equations imply equal matrices, grounded in core linear algebra principles. It supports learners and practitioners in avoiding common pitfalls and correctly applying invertibility concepts in their work.

Frequently Asked Questions

If AB = AC and A is invertible, can we conclude that B = C? Why or why not?

Yes, if A is invertible and AB = AC, then multiplying both sides by A^{-1} yields B = C. This is because invertibility allows us to 'cancel' A from both sides.

Given that A, B, and C are N x P matrices with A invertible, does the equation AB = AC imply B = C in general?

Yes, since A is an invertible N x N matrix, AB = AC implies B = C by multiplying both sides with A^{-1} on the left.

Does the statement 'Suppose AB = AC and A is invertible' hold for non-square matrices? Why or why not?

The statement holds only if A is a square invertible matrix. For non-square matrices, invertibility may not hold, and the conclusion B = C may not follow.

In the case where A is invertible and AB = AC, can we deduce that B = C even if B and C are not square matrices?

Yes, provided A is square (and invertible), then B = C follows directly. If B and C are not square, invertibility of A must be defined accordingly, and the conclusion still holds.

Is the statement 'Suppose AB = AC, where A is invertible' always true for matrices over complex numbers? Why?

Yes, over complex numbers, as long as A is invertible, the property that AB = AC implies B = C holds, due to the existence of A^{-1}.

What is the general principle demonstrated by the statement 'If AB = AC and A is invertible, then B = C'?

The principle is that invertible matrices act as invertible transformations, allowing us to cancel them from equations and establish equality of the other matrices involved.

Additional Resources

Suppose AB = AC, where A and C are $N\times p$ matrices and A is invertible. Show that B = C. Is this true in general?
This question touches on fundamental concepts in linear algebra concerning the properties of matrices, especially regarding invertibility and the implications of matrix equations. Understanding whether the equality AB = AC implies B = C under certain conditions is a key aspect of matrix theory that has wide-ranging applications in systems of linear equations, transformations, and more. In this article, we will explore this problem in detail, analyze the conditions under which the statement holds, and discuss its limitations and generalizations.

Understanding the Problem

Let's first restate the problem in our own words:

- You are given matrices A, B, and C.
- Matrices A and C are both \(N \times p \) matrices.
- Matrix A is invertible, meaning A is a square matrix (\(N \times N \)) and has a non-zero determinant, allowing for the existence of A\(^{-1}\).
- The equation \(AB = AC \) holds, where B and C are also \(N \times p \) matrices.

The question is: Does \(AB=AC \) imply \(B=C \)? Specifically, the goal is to determine whether B = C in this context and whether this conclusion is valid in general.

Clarifying the Dimensions and Notation

Before diving into the proof or counterexamples, ensure clarity of the matrix dimensions:

- A: \(N \times N \) (square and invertible)
- B: \(N \times p \)
- C: \(N \times p \)

Given this, the product \(AB \) and \(AC \) are both \(N \times p \) matrices.

The Core Question: Does \(AB=AC \Rightarrow B=C \)?

This is a classic question related to the left-cancellation property in matrix multiplication.

When is cancellation valid?

- In general, for matrices over a field, if \(A\) is invertible, then multiplication by \(A^{-1}\) from the left preserves equality, i.e.,

\[

$$
AB = AC \Rightarrow A^{-1}AB = A^{-1}AC \Rightarrow B= C
$$

- Key point: The invertibility of A is crucial here.

Formal Proof Under the Assumption that A is Invertible

Let's carefully demonstrate that:

If A is invertible and $AB = AC$, then $B = C$.

Step-by-step proof:

1. Start with the given:

$$
AB = AC
$$

2. Multiply both sides from the left by A^{-1}:

$$
A^{-1}AB = A^{-1}AC
$$

3. Simplify:

$$
(I) B = (I) C
$$

where I is the $N \times N$ identity matrix.

4. Result:

$$
B = C
$$

Conclusion: When A is invertible, the equality $AB=AC$ implies $B=C$.

Is the Statement True in General?

The answer hinges on the invertibility of A. Let's explore the possibilities:

1. When A is invertible

- As shown above, yes, the statement holds: $AB=AC \Rightarrow B=C$.

2. When A is not invertible

- The statement may not hold in general.

- Counterexample:

Suppose A is a singular matrix (non-invertible). Then, there exists a non-zero matrix X such that:

$$
A X = 0
$$

- This implies the existence of non-trivial matrices X with:

$$
A(B - C) = 0
$$

- Therefore, if A is not invertible, then from $AB = AC$, we cannot necessarily conclude that $B = C$. The difference $B - C$ might lie in the null space of A:

$$
A(B - C) = 0
$$

- Hence, in the non-invertible case, the conclusion does not necessarily hold.

Formal Statement and Explanation

Theorem:

Let A be an $N \times N$ matrix over a field (e.g., real numbers), and B,C be $N \times p$ matrices. Then:

$$
AB = AC \quad \text{and} \quad A \text{ is invertible} \quad \Rightarrow \quad B = C
$$

Conversely:

- If A is not invertible, then $AB=AC$ does not necessarily imply $B=C$.

Intuition:

- When A is invertible, it is left-cancellable; you can "divide" both sides by A on the left.

- When A is singular, this "division" isn't possible, and the null space of A can contain non-zero matrices $X = B - C$ such that $A X = 0$.

Practical Implications and Applications

Understanding this property is essential in many areas:

- Linear systems: Ensuring the uniqueness of solutions relies on invertibility.

- Transformations: When applying linear transformations represented by matrices, invertibility guarantees that the original vector or matrix can be recovered.

- Data science and machine learning: Matrix equations involving invertible matrices are used in solving regression problems, invertible transformations, and more.

Summary list:

- If A is invertible:
$AB = AC \Rightarrow B = C$

- If A is not invertible:
$AB=AC$ does not necessarily imply $B=C$. Additional conditions are required.

Additional Considerations and Generalizations

1. Left vs. Right Cancellation

- The above discussion focuses on left cancellation: multiplying from the left by an invertible matrix.

- Right cancellation (multiplying from the right) requires B and C to be square and invertible or additional conditions.

2. Matrices over different fields

- The same logic applies over any field where matrices are defined, such as real numbers, complex numbers, etc.

3. Generalizations to non-invertible matrices

- In the case where A is not invertible, the set of solutions to $AB=AC$ includes matrices B and C with:

$$
A(B - C) = 0
$$

- The set of solutions forms an affine space, and the difference \(B - C\) lies in the null space of the linear transformation defined by \(A\).

Summary and Final Remarks

- Key takeaway: The statement "Suppose \(AB=AC\), where \(A\) and \(C\) are \(N \times p\) matrices and \(A\) is invertible, then \(B=C\)" is true due to the invertibility of \(A\).

- In general, without the invertibility of \(A\), the statement does not hold, and counterexamples exist.

- This highlights the importance of invertibility in linear algebra when it comes to the uniqueness of solutions and the cancellation property.

Final Thoughts

Understanding the conditions under which matrix equations imply equality of matrices is fundamental for solving systems efficiently and analyzing transformations. The invertibility of matrices plays a central role in enabling cancellation properties that simplify complex problems, making them easier to analyze and interpret. Recognizing when these properties hold—and when they do not—is crucial for both theoretical insights and practical applications across science and engineering disciplines.

In conclusion:
Yes, in the case where \(A\) is invertible, the equality \(AB=AC\) does imply \(B=C\). However, this is not true in general if \(A\) is singular, emphasizing the critical role of invertibility in matrix algebra.

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