Suppose That We Roll A Fair Die Until A 6 Comes Up Or We Have Rolled It 10 Times. What Is The Expected

Suppose That We Roll A Fair Die Until A 6 Comes Up Or We Have Rolled It 10 Times. What Is The Expected

Understanding the expected value in probabilistic scenarios is fundamental in probability theory and statistics. When dealing with sequences of random events, such as rolling dice, the concept of expectation helps quantify the average outcome over numerous repetitions. In this article, we explore a specific problem involving rolling a fair six-sided die repeatedly until either a 6 appears or we reach the 10th roll. Our goal is to determine the expected number of rolls under these conditions.

Problem Restatement and Key Concepts

Restating the Problem

Imagine the following scenario: you have a fair six-sided die, with faces numbered 1 through 6. You roll the die repeatedly, and you stop under one of two conditions:
- When a 6 appears for the first time.
- When you have completed 10 rolls, whichever comes first.

The question posed is: What is the expected number of rolls performed under this stopping rule?

Key Concepts Involved

To analyze this problem, several concepts from probability theory are necessary:
- Expected Value (E): The average number of rolls we can anticipate, over many repetitions of the experiment.
- Stopping Time: The random variable representing the number of rolls until the stopping condition (a 6 appearing or reaching 10 rolls).
- Geometric Distribution: The distribution modeling the number of trials until the first success, which in this case is rolling a 6.
- Conditional Probability: Probabilities conditioned on previous outcomes and the stopping rule.

Modeling the Problem

Defining the Random Variable

Let T be the random variable denoting the total number of rolls performed until the process stops. Our goal is to compute:

$$
E[T]
$$

which is the expected value of T.

Possible Values of T

The process terminates at:
- The first roll where a 6 appears, which can occur at any roll t where $1 \leq t \leq 10$.
- Or, if no 6 appears in the first 10 rolls, the process stops at $t=10$.

Thus, T can take values in $\{1, 2, 3, \dots, 10\}$.

Breaking Down the Problem

The expected value can be expressed as:

$$
E[T] = \sum_{t=1}^{10} t \times P(T = t)
$$

where $P(T = t)$ is the probability that the process stops exactly at the t-th roll.

Calculating the Probabilities $P(T = t)$

Probability That the First $t-1$ Rolls Are Not 6, And the t-th Is 6

For $1 \leq t \leq 10$, the process stops at roll t if:
- The first $t-1$ rolls do not show a 6.
- The t-th roll is a 6.

Since the die is fair:
- The probability that a single roll is not a 6 is $\frac{5}{6}$.
- The probability that a roll is a 6 is $\frac{1}{6}$.

Therefore:

$$
P(T = t) = \left(\frac{5}{6} \right)^{t-1} \times \frac{1}{6}
$$

for $t = 1, 2, \dots, 10$.

Probability That No 6 Occurs in 10 Rolls

If no 6 appears in the first 10 rolls:

$$
P(T > 10) = \left(\frac{5}{6} \right)^{10}
$$

but since the process stops at 10 regardless, this scenario corresponds to $T=10$ with the caveat that no 6 appeared in the first 10 rolls.

However, because the process stops at 10 if no 6 has appeared, the probability of stopping exactly at 10 is:

$$
P(T=10) = \left(\frac{5}{6} \right)^{9} \times \frac{1}{6} \quad \text{(if a 6 appears on 10th roll)}
$$

but we need to consider the case where no 6 appears in all 10 rolls. Since the stop is at 10 when no 6 appears, the probability that the process stops at $T=10$ because no 6 has appeared in 10 rolls is:

$$
P(T=10) = \left(\frac{5}{6} \right)^{10}
$$

but this overlaps with the previous calculation. To keep consistent, we realize that:
- For $t=1, \dots, 9$, $P(T=t)$ is as above.
- For $t=10$, the process stops either because:
- A 6 appears on the 10th roll, with probability $\left(\frac{5}{6} \right)^9 \times \frac{1}{6}$.
- Or no 6 appears in all 10 rolls, with probability $\left(\frac{5}{6} \right)^{10}$.

But since the process stops at 10 regardless, the total probability that $T=10$ is:

$$
P(T=10) = \left(\frac{5}{6} \right)^9 \times \frac{1}{6} + \left(\frac{5}{6} \right)^{10} = \left(\frac{5}{6} \right)^9 \times \frac{1}{6} + \left(\frac{5}{6} \right)^{10}
$$

Alternatively, note that the probability that the process stops at exactly 10 because no 6 has appeared in the first 10 rolls is $\left(\frac{5}{6} \right)^{10}$. The probability that it stops at 10 because a 6 appears on the 10th roll is $\left(\frac{5}{6} \right)^9 \times \frac{1}{6}$.

Thus, the total probability that the process stops at 10 is:

$$
P(T=10) = \left(\frac{5}{6} \right)^9 \times \frac{1}{6} + \left(\frac{5}{6} \right)^{10}
$$

Calculating the Expected Value $E[T]$

Putting It All Together

The expected value is:

$$
E[T] = \sum_{t=1}^{9} t \times P(T=t) + 10 \times P(T=10)
$$

Substituting the probabilities:

$$
E[T] = \sum_{t=1}^{9} t \times \left(\frac{5}{6} \right)^{t-1} \times \frac{1}{6} + 10 \times \left[\left(\frac{5}{6} \right)^9 \times \frac{1}{6} + \left(\frac{5}{6} \right)^{10} \right]
$$

Simplifying the Calculation

Expected Value of a Geometric Distribution with Truncation

The scenario resembles a geometric distribution truncated at 10, with the "success" being rolling a 6. The standard expectation of a geometric distribution (with success probability $p = \frac{1}{6}$) is:

$$
E[\text{number of trials until first success}] = \frac{1}{p} = 6
$$

However, because the process is truncated at 10, the actual expectation differs.

Using the Law of Total Expectation

An alternative approach involves considering the probability that a 6 appears at each roll and the chance that it doesn't appear at all in 10 rolls.

The expectation can be viewed as:
\[
E[T] = \sum_{t=1}^{10} t \times P(\text{first success at } t) + 10 \times P(\text{no success in 10 trials})
\]

Given the probabilities:
- \(P(\text{success at } t) = \left(\frac{5}{6} \right)^{t-1} \times \frac{1}{6} \),
- \(P(\text{no success in 10 trials}) = \left(\frac{5}{6} \right)^{10} \).

So, the expected value simplifies to:
\[
E[T] = \sum_{t=1}^{9} t \times \left(\frac{5}{6} \right)^{t-1} \times \frac{1}{6} + 10 \times \left(\frac{5}{6} \right)^{10

Frequently Asked Questions

What is the probability that we roll a 6 on the first try when rolling a fair die?

The probability is 1/6 since there is one 6 and six possible outcomes.

How do we model the process of rolling a die until a 6 appears or we have rolled 10 times?

This process can be modeled as a geometric or negative binomial experiment with a maximum of 10 trials, where success is rolling a 6.

What is the expected number of rolls until a 6 appears if we do not have a maximum of 10 rolls?

For an infinite process, the expected number of rolls is 6, since each roll has a 1/6 chance of success.

How do we compute the expected number of rolls in this truncated process with a maximum of 10 rolls?

We consider the probability of success at each roll and sum over the possible outcomes, taking into account the stopping rule at 10 rolls.

What is the probability that a 6 appears within the first n rolls?

The probability is 1 minus the probability that no 6 appears in the first n rolls, which is $(5/6)^n$.

How does the maximum of 10 rolls affect the expected number of rolls until a 6 appears?

The maximum cap means the expected number is less than or equal to the expected value in an unlimited process, adjusted for the probability that a 6 appears before or at the 10th roll.

What is the probability that we stop after exactly n rolls due to getting a 6 on the nth roll, with n ≤ 10?

It is $(5/6)^{n-1} (1/6)$, representing no 6 in previous rolls and a 6 on the nth roll.

How do we calculate the overall expected number of rolls in this scenario?

The expected value is the sum over n=1 to 10 of n times the probability that the process stops at exactly n, plus the probability that no 6 appears in all 10 rolls multiplied by 10.

What is the probability that we do not get a 6 in 10 rolls?

The probability is $(5/6)^{10}$, since each roll has a 5/6 chance of not being a 6.

Additional Resources

Expected Value in a Limited Roll Scenario: An In-Depth Analysis

When it comes to understanding probability and expected outcomes, one of the most fundamental yet intriguing problems involves rolling a fair die repeatedly until a certain event occurs—such as rolling a 6—or until a predetermined number of rolls have been completed. This scenario exemplifies key concepts in probability theory, including geometric distributions, expected values, and the influence of constraints on random processes.

In this comprehensive review, we analyze the problem:

Suppose that we roll a fair six-sided die repeatedly until either a 6 appears or we have rolled the die 10 times. What is the expected number of rolls needed—i.e., the expected number of rolls until the stopping condition?

This question, simple in its phrasing, opens a window into the elegant interplay between chance, expectation, and constraints. Our goal is to explore this problem thoroughly, providing insights into the underlying mathematics, assumptions, and practical implications.

Understanding the Scenario: The Core Components

Before delving into calculations, it's essential to clarify the key elements of the problem:

1. The Die and Its Fairness

The die is a standard six-sided object with faces numbered 1 through 6. Since it is fair, each face has an equal probability of:

$$ P(\text{any face}) = \frac{1}{6} $$

2. The Stopping Conditions

- Condition A: A 6 appears on the current roll.
- Condition B: The total number of rolls reaches 10.

The process stops as soon as either condition is satisfied. This introduces a truncated or capped process: the maximum number of rolls is 10.

3. The Random Variable

Let T denote the stopping time, i.e., the number of rolls performed until the process halts.

Our goal is to compute:

$$ \mathbb{E}[T] $$

the expected value of T.

Key Probabilistic Insights and Approach

This problem can be viewed as a stopping time problem in probabilistic processes, specifically a variation of the geometric distribution with an upper limit.

1. The Geometric Distribution

In an unbounded setting—rolling until a 6 appears—the number of rolls needed follows a geometric distribution with success probability:

$$ p = \frac{1}{6} $$

The expected number of rolls until the first success (a 6) is:

$$ \mathbb{E}[T_{\text{geometric}}] = \frac{1}{p} = 6 $$

but because our process is capped at 10 rolls, the distribution is truncated.

2. Incorporating the Cap

Since we stop at 10 rolls regardless of whether a 6 appears or not, the process becomes a truncated geometric process. The key question becomes:

- What is the probability that a 6 occurs at each roll before reaching the cap?
- How does the cap influence the expected number of rolls?

Step-by-Step Calculation of the Expected Value

To rigorously compute $\mathbb{E}[T]$, consider the following approach:

1. Define the Stopping Probability at Each Roll

- The probability that the process stops exactly at the k-th roll is:

$$
P(T = k) = P(\text{no 6 in first } k-1 \text{ rolls}) \times P(\text{6 on the }k\text{-th roll})
$$

for $1 \leq k \leq 9$, because if a 6 appears at any of these rolls, the process stops immediately.

- For the 10-th roll, the process stops either because a 6 appears or because we've reached the maximum number of rolls.

2. Calculating the Probability of Success at each Step

- The probability that the first success occurs at the k-th roll:

$$
P(T = k) = (1 - p)^{k-1} p

where $p = 1/6$. This is the geometric distribution.

- For the first 9 rolls:

$$
P(T = k) = (1 - \frac{1}{6})^{k-1} \times \frac{1}{6}
$$

- For the 10th roll, however, the process may have not yet seen a 6. The probability that the process reaches the 10th roll without success is:

$$
(1 - p)^9
$$

and at the 10th roll:

- The process stops because either:

a) A 6 appears at the 10th roll, with probability $p = 1/6$.

b) No 6 appears in all 10 rolls—meaning the process ends with no success (which is not a stopping condition, but the process terminates anyway).

3. Computing the Expected Value

The expected number of rolls is:

$$
\mathbb{E}[T] = \sum_{k=1}^{10} k \times P(T = k)
$$

where:

- For $1 \leq k \leq 9$:

$$
P(T = k) = (1 - p)^{k-1} p
$$

- For $k = 10$, the process stops because:

- Either a 6 appears on the 10th roll:

$$
P(\text{6 at 10th roll} \text{ and no earlier 6}) = (1 - p)^9 \times p
$$

- Or no 6 appears in 10 rolls:

$$
(1 - p)^{10}
$$

But since the process stops at 10 regardless, the probability that the process stops exactly at 10th roll (either with or without success) is:

$$
P(T=10) = (1 - p)^9 \times p + (1 - p)^{10}
$$

However, in the original problem, the process stops at the 10th roll regardless, so the probability of stopping exactly at 10th roll is:

$$
P(T=10) = (1 - p)^9 + (1 - p)^9 \times p
$$

which simplifies to:

$$
P(T=10) = (1 - p)^9 (1 + p)
$$

But this overcounts the scenario where no success occurs at all, which is acceptable because the process terminates anyway.

Explicit Calculation of the Expected Value

Now, let's formalize the calculations.

1. Probability of success at each early roll (1 to 9):

$$
P(T = k) = (1 - p)^{k-1} p, \quad 1 \leq k \leq 9
$$

2. Probability that no success occurs in the first 9 rolls:

$$
P(\text{no success in first 9}) = (1 - p)^9
$$

3. Probability that success occurs exactly at the 10th roll:

$$
P(T=10) = (1 - p)^9 \times p

$$

because the process reaches the 10th roll without success, and then success occurs at the 10th.

4. Probability that no success occurs at all:

$$
P(\text{no success in 10 rolls}) = (1 - p)^{10}
$$

which is the probability that no 6 appears in the first 10 rolls.

5. Calculating the expected value:

$$
\mathbb{E}[T] = \sum_{k=1}^9 k \times P(T=k) + 10 \times P(T=10 \text{ or no success})
$$

But since the process always stops at or before 10 rolls:

$$
\mathbb{E}[T] = \sum_{k=1}^9 k \times (1 - p)^{k-1} p + 10 \times \left((1 - p)^9 p + (1 - p)^{10} \right)
$$

which simplifies to:

$$
\boxed{
\mathbb{E}[T] = p \sum_{k=1}^9 k (1 - p)^{k-1} + 10 \left((1 - p)^9 p + (1 - p)^{10} \right)
}
$$

Numerical Evaluation and Interpretation

Using the specific probability $p = \frac{1}{6}$, we can now compute the expected value explicitly.

1. Define parameters:

$$
p = \frac{1}{6} \approx 0.1667
$$

$$
q = 1 - p = \frac{5}{6} \approx 0.8333
$$

2. Compute the sum:

$$
\sum_{k=1}^9 k q^{k-1}
$$

This is a finite sum involving a geometric series with a weighted term.

Recall that:

$$
\sum_{k=1}^{n} k r^{k-1} = \frac{1 - (n+1) r^{n} + n r^{n+1}}{(1 - r)^2}
$$

Suppose That We Roll A Fair Die Until A 6 Comes Up Or We Have Rolled It 10 Times What Is The Expected

Suppose That We Roll A Fair Die Until A 6 Comes Up Or We Have Rolled It 10 Times What Is The Expected

Related Articles

- This Is Your Code.>>> A = [5, 10, 15]>>> B = [2, 4, 6]>>> C = ['dog','cat',
- The Car Drove Down The Street. Next, The Teacher Asks The Students To Rewrite This Sentence, Providing
- The Sales Tax Rate Is 6%. If You Paid $48 In Sales Tax For A Lawn Mower, What Was The Cost Of The Lawn

Back to Home