

The Auxiliary Equation For The Given Differential Equation Has Complex Roots. Find A General Solution.

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Understanding how to solve differential equations is fundamental in many fields such as engineering, physics, and mathematics. When dealing with linear homogeneous differential equations with constant coefficients, one of the key steps involves analyzing the auxiliary (or characteristic) equation. In particular, the nature of the roots of this auxiliary equation—whether real and distinct, repeated, or complex—dictates the form of the general solution. This article focuses on the scenario where the auxiliary equation has complex roots, guiding you through the process of finding the general solution, interpreting the roots, and applying the method to various examples.

Understanding the Auxiliary Equation

Before delving into the specifics of complex roots, it is essential to understand what the auxiliary equation is and how it relates to differential equations.

Definition of the Auxiliary Equation

For a linear homogeneous differential equation with constant coefficients, such as:

$$\left[a \frac{d^n y}{dx^n} + b \frac{d^{n-1} y}{dx^{n-1}} + \dots + c y = 0, \right]$$

the auxiliary (or characteristic) equation is obtained by replacing derivatives with powers of a variable (r) :

$$\left[a r^n + b r^{n-1} + \dots + c = 0. \right]$$

The solutions (r) to this algebraic equation are called roots and are crucial in constructing the general solution.

Significance of Roots of the Auxiliary Equation

- Real and distinct roots: Lead to exponential solutions.
- Repeated roots: Lead to solutions multiplied by powers of (x) .
- Complex roots: Lead to solutions involving sinusoidal functions combined with exponentials.

When the roots are complex, they typically come in conjugate pairs, which significantly influence the form of the general solution.

Complex Roots of the Auxiliary Equation

When solving the auxiliary equation, encountering complex roots is common, especially when the differential equation models oscillatory phenomena such as vibrations or electrical circuits.

Form of Complex Roots

Suppose the auxiliary equation yields roots:

$$r = \alpha \pm \beta i,$$

where:

- α is the real part,
- β is the imaginary part.

These roots are conjugate pairs because the coefficients of the differential equation are real.

Implications for the General Solution

The general solution corresponding to these roots is:

$$y(x) = e^{\alpha x} \left(C_1 \cos \beta x + C_2 \sin \beta x \right),$$

where C_1 and C_2 are arbitrary constants.

This form combines exponential decay or growth (depending on α) with oscillatory behavior governed by sine and cosine functions.

Deriving the General Solution for Complex Roots

The process of deriving the general solution involves several steps, which are outlined below.

Step 1: Find the Auxiliary Equation

- Convert the differential equation into its characteristic polynomial form.
- Solve for roots λ .

Step 2: Analyze the Roots

- Determine whether roots are real, repeated, or complex.
- For complex roots, identify the real part α and imaginary part β .

Step 3: Write the General Solution

- Use the conjugate root theorem to write the solution involving exponentials and trigonometric functions:

$$y(x) = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$$

This form ensures that the solution is real-valued, which is necessary for most physical applications.

Step 4: Apply Initial Conditions (if available)

- Use initial or boundary conditions to find specific values of (C_1) and (C_2) .

Examples of Differential Equations with Complex Roots

To illustrate the method, let's examine some common differential equations with complex roots and derive their solutions.

Example 1: Second-Order Homogeneous Differential Equation

Solve:

$$[y'' + 4y' + 13y = 0.]$$

Solution:

1. Auxiliary equation:

$$[r^2 + 4r + 13 = 0.]$$

2. Roots:

$$[r = \frac{-4 \pm \sqrt{16 - 52}}{2} = \frac{-4 \pm \sqrt{-36}}{2} = \frac{-4 \pm 6i}{2} = -2 \pm 3i.]$$

3. General solution:

$$[y(x) = e^{-2x} \left(C_1 \cos 3x + C_2 \sin 3x \right).]$$

Example 2: Higher-Order Differential Equation

Solve:

$$[y''' + 2 y'' + y' + 0 y = 0.]$$

Solution:

1. Auxiliary equation:

$$[r^3 + 2 r^2 + r = 0.]$$

2. Factor:

$$[r (r^2 + 2 r + 1) = 0.]$$

3. Roots:

$$- (r = 0),$$

$$- (r^2 + 2 r + 1 = 0 \Rightarrow (r + 1)^2 = 0 \Rightarrow r = -1) \text{ (double root).}$$

Since all roots are real, but suppose we modify the equation to have complex roots, for example:

$$[y'' + 2 y' + 5 y = 0.]$$

- Auxiliary equation:

$$[r^2 + 2 r + 5 = 0.]$$

- Roots:

$$r = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i.$$

- General solution:

$$y(x) = e^{-x} \left(C_1 \cos 2x + C_2 \sin 2x \right).$$

Applications of the Solution with Complex Roots

The solutions involving exponential and sinusoidal functions are prevalent across various disciplines.

Physics and Engineering

- Mechanical vibrations where damping causes oscillations modeled by solutions with complex roots.
- Electrical circuits involving RLC components often lead to differential equations with complex roots, representing oscillatory current or voltage.

Biology and Ecology

- Population models with oscillatory behavior due to feedback mechanisms.

Mathematical Analysis and Signal Processing

- Fourier analysis and filter design often involve exponential and sinusoidal components derived from differential equations with complex roots.

Summary and Key Takeaways

- The auxiliary equation of a differential equation determines the form of the general solution.
- When the roots are complex conjugates $(\alpha \pm \beta i)$, the general solution involves exponential decay/growth modulated by sinusoidal functions.
- The general solution takes the form:

$$y(x) = e^{\alpha x} \left(C_1 \cos \beta x + C_2 \sin \beta x \right).$$

- This form captures oscillatory phenomena, which are widespread in natural and engineered systems.

Final Notes

Mastering the process of solving differential equations with complex roots is essential for analyzing systems exhibiting oscillatory or wave-like behavior. Recognizing the roots of the auxiliary equation and translating them into the appropriate solution form enables you to model real-world phenomena accurately. Always verify the roots and apply initial conditions carefully to obtain particular solutions suited to your specific problem.

Disclaimer: This article provides a comprehensive overview of solving linear homogeneous differential equations with complex roots. For more complex or higher-order cases, additional techniques such as variation of parameters or undetermined coefficients may be required.

Frequently Asked Questions

What does it mean when the auxiliary equation of a differential equation has complex roots?

When the auxiliary equation has complex roots, it indicates that the solution to the differential equation involves oscillatory functions, typically sines and cosines, reflecting a system with oscillatory behavior.

How do you find the general solution of a differential equation when the auxiliary equation has complex conjugate roots?

If the auxiliary equation has complex conjugate roots $\alpha \pm \beta i$, the general solution is expressed as $y(t) = e^{\alpha t} [C_1 \cos(\beta t) + C_2 \sin(\beta t)]$, where C_1 and C_2 are arbitrary constants.

Why do complex roots of the auxiliary equation lead to solutions involving sine and cosine functions?

Complex roots have the form $\alpha \pm \beta i$, and by Euler's formula, $e^{(\alpha \pm \beta i)t}$ can be written as $e^{\alpha t} (\cos \beta t \pm i \sin \beta t)$. Since solutions must be real, the real and imaginary parts give rise to sine and cosine functions in the general solution.

Can you provide an example of solving a differential equation with

complex roots in its auxiliary equation?

Yes. For example, consider $y'' + 4y' + 13y = 0$. The auxiliary equation is $r^2 + 4r + 13 = 0$, which has roots $r = -2 \pm 3i$. The general solution is $y(t) = e^{-2t} [C_1 \cos 3t + C_2 \sin 3t]$.

What steps should I follow to solve a differential equation with complex roots in the auxiliary equation?

First, find the auxiliary equation and solve for its roots. If roots are complex conjugates $\alpha \pm \beta i$, then write the general solution as $y(t) = e^{\alpha t} [C_1 \cos(\beta t) + C_2 \sin(\beta t)]$. Determine the constants using initial conditions if available.

Additional Resources

The Auxiliary Equation For The Given Differential Equation Has Complex Roots. Find A General Solution.

Understanding how to solve differential equations is fundamental in many fields of science, engineering, and mathematics. When tackling second-order linear differential equations with constant coefficients, the auxiliary (or characteristic) equation plays a pivotal role in determining the form of the general solution. A particularly interesting case arises when the roots of this auxiliary equation are complex conjugates. This scenario introduces unique features into the solution structure, demanding a careful approach to derive the most general form. In this article, we will explore this situation in detail, providing insights into the process, the underlying theory, and practical considerations.

Introduction to Differential Equations and Auxiliary Equations

Differential equations describe how a particular quantity changes concerning another. Among the simplest and most studied are second-order linear differential equations with constant coefficients, typically expressed as:

$$[a y'' + b y' + c y = 0]$$

where (a) , (b) , and (c) are constants, and (y'') and (y') denote the second and first derivatives of (y) with respect to the independent variable, often (t) or (x) .

To solve such equations, we often assume solutions of the form:

$$[y = e^{r t}]$$

Substituting into the differential equation leads to the auxiliary (or characteristic) equation:

$$[a r^2 + b r + c = 0]$$

The roots of this quadratic determine the nature of the general solution.

The Nature of Roots of the Auxiliary Equation

The roots of the auxiliary equation are found using the quadratic formula:

$$[r = \frac{-b \pm \sqrt{b^2 - 4 a c}}{2a}]$$

Depending on the discriminant $(D = b^2 - 4 a c)$, roots can be classified as:

- Real and Distinct Roots: $(D > 0)$

- Repeated Roots: $(D = 0)$
- Complex Conjugate Roots: $(D < 0)$

This article focuses on the last case, where the roots are complex conjugates, i.e.,

$$r = \alpha \pm i \beta$$

with $(\alpha, \beta \in \mathbb{R}), (\beta \neq 0)$.

Complex Roots: Derivation and Characteristics

When the discriminant is negative, the roots become:

$$r = \frac{-b}{2a} \pm i \frac{\sqrt{4ac - b^2}}{2a}$$

Set:

$$\alpha = \frac{-b}{2a} \quad \text{and} \quad \beta = \frac{\sqrt{4ac - b^2}}{2a}$$

The general solution corresponding to these roots leverages Euler's formula:

$$e^{\alpha t} (\cos \beta t + i \sin \beta t)$$

Since the differential equation has real coefficients, the real solutions are constructed from the real and imaginary parts of the complex exponential solutions, leading to a real-valued general solution.

Constructing the General Solution for Complex Roots

The key idea is that the general solution for the differential equation when roots are complex conjugates takes the form:

$$y(t) = e^{\alpha t} (C_1 \cos \beta t + C_2 \sin \beta t)$$

where C_1 and C_2 are arbitrary real constants.

Derivation process:

1. Start with the roots:

$$r = \alpha \pm i \beta$$

2. The solutions corresponding to each root are:

$$y_1(t) = e^{(\alpha + i \beta) t}$$

$$y_2(t) = e^{(\alpha - i \beta) t}$$

3. Use Euler's formula:

$$e^{i \beta t} = \cos \beta t + i \sin \beta t$$

4. Real and imaginary parts of these exponential functions give two linearly independent solutions:

$$y_1(t) = e^{\alpha t} \cos \beta t$$

$$y_2(t) = e^{\alpha t} \sin \beta t$$

5. The general solution is then:

$$y(t) = C_1 e^{\alpha t} \cos \beta t + C_2 e^{\alpha t} \sin \beta t$$

Features and Applications

Features:

- The exponential term $(e^{\alpha t})$ governs the growth or decay depending on the sign of (α) .
- The sine and cosine terms introduce oscillatory behavior due to the imaginary part (β) .
- The solution is valid for all real (t) , capturing damping or amplification combined with oscillations.

Applications:

- Mechanical vibrations with damping (e.g., a mass-spring system with friction).
- Electrical circuits involving RLC components.
- Quantum mechanics and wave phenomena.
- Population dynamics with oscillatory components.

Advantages and Limitations

Advantages:

- Provides a clear, systematic way to find solutions for second-order linear differential equations.
- The method directly relates the roots of the auxiliary equation to the behavior of the solution.
- The form $(e^{\alpha t} (\text{sin or cos}))$ is intuitive for oscillatory and damping phenomena.

Limitations:

- Requires solving quadratic equations; complex roots only occur when the discriminant is negative.
- The method assumes constant coefficients; variable coefficient equations require different approaches.
- Interpreting parameters α and β in physical systems may require additional context.

Step-by-step Example

Let's illustrate the process with a concrete example:

Given differential equation:

$$y'' + 4y' + 13y = 0$$

Step 1: Write the auxiliary equation:

$$r^2 + 4r + 13 = 0$$

Step 2: Calculate the discriminant:

$$D = 4^2 - 4 \times 1 \times 13 = 16 - 52 = -36$$

Since $(D < 0)$, roots are complex conjugates.

Step 3: Find α and β :

$$\alpha = \frac{-b}{2a} = \frac{-4}{2} = -2$$

$$\beta = \frac{\sqrt{52 - 16}}{2} = \frac{\sqrt{36}}{2} = \frac{6}{2} = 3$$

Step 4: Write the general solution:

$$y(t) = e^{-2t} (C_1 \cos 3t + C_2 \sin 3t)$$

This solution describes oscillations with an exponential decay, typical in damping systems.

Conclusion and Final Remarks

When the auxiliary equation of a second-order linear differential equation has complex roots, the solution naturally involves exponential functions multiplied by sinusoidal functions. This combination captures oscillatory behavior modulated by exponential growth or decay, depending on the sign of the real part (α). Recognizing the nature of roots early simplifies the solving process and provides insights into the qualitative behavior of the system modeled by the differential equation.

Understanding this methodology is crucial for students and practitioners, as it bridges algebraic roots with physical phenomena like oscillations, damping, and resonance. Mastery of deriving the general solution from complex roots enhances one's ability to analyze and interpret a wide range of dynamic systems.

In summary:

- The auxiliary equation roots dictate the form of the solution.
- Complex roots lead to solutions involving exponentials times sine and cosine.
- The general solution provides a complete description of the system's behavior.
- Recognizing the pattern is essential in fields from engineering to physics.

By mastering these concepts, one gains a powerful tool to solve and interpret second-order linear differential equations with complex roots, enriching both theoretical understanding and practical problem-solving skills.

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