The Curve Y=ax^(2)+bx+c Passes Through The Point (2,28) And Is Tangent To The Line Y=4x At The Origin.

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Understanding the properties and equations of quadratic curves is essential in various fields such as mathematics, physics, engineering, and computer graphics. In this article, we explore a specific quadratic curve defined by the equation $Y = ax^2 + bx + c$, which satisfies two key conditions: it passes through the point (2, 28), and it is tangent to the line Y = 4x at the origin. We will delve into the mathematical process of determining the values of the coefficients a, b, and c, analyze the geometric implications, and discuss applications of such quadratic curves.

Understanding the Problem: Key Conditions and Their Significance

Before solving for the coefficients, it's important to understand the conditions given and what they imply about the curve.

Passing Through the Point (2, 28)

- The quadratic curve must satisfy the point (2, 28), meaning when x = 2, Y = 28.
- Mathematically, this gives us the equation:

$a(2)^2 + b(2) + c = 28$

- Simplifies to:

$4a + 2b + c = 28$

Tangent to the Line Y = 4x at the Origin

- The line Y = 4x touches the quadratic curve at the origin (0, 0), and their slopes at that point are equal.
- This implies two conditions:

1. The curve passes through the origin:

$a(0)^2 + b(0) + c = 0 \Rightarrow c = 0$

2. The slope of the tangent to the quadratic at x = 0 equals the slope of the line Y = 4x, which is 4.

- The derivative of the quadratic:

$\frac{dY}{dx} = 2ax + b$

- At x = 0:

$\frac{dY}{dx} = b$

- Set equal to 4:

$b = 4$

Deriving the Coefficients a, b, and c

With the above conditions, we now have two key equations:

1. $c = 0$
2. $b = 4$

Substituting these into the passing through (2, 28) condition:

$$
4a + 2b + c = 28
$$

becomes

$$
4a + 2(4) + 0 = 28
$$

which simplifies to:

$$
4a + 8 = 28
$$

Solving for a:

$$
4a = 20
\Rightarrow a = 5
$$

Therefore, the quadratic equation is:

$$
Y = 5x^2 + 4x + 0
$$

or simply:

$$
Y = 5x^2 + 4x
$$

Verifying the Conditions

To ensure our solution is correct, verify the two key conditions:

Check Passage Through (2, 28)

Substitute $x = 2$:

$$
Y = 5(2)^2 + 4(2) = 5(4) + 8 = 20 + 8 = 28
$$

which satisfies the point condition.

Check Tangency at the Origin

- At $x = 0$:

$$
Y = 5(0)^2 + 4(0) = 0
$$

which confirms the curve passes through (0, 0).

- Slope at $x = 0$:

$$
\frac{dY}{dx} = 2a x + b = 2(5)(0) + 4 = 4
$$

matching the slope of the line $Y = 4x$.

All conditions are satisfied, confirming the quadratic function:

$$
Y = 5x^2 + 4x
$$

Graphical Interpretation and Geometric Insights

Visualizing the quadratic curve and its relation to the line $Y = 4x$ provides deeper understanding.

Shape and Position of the Curve

- Since $a = 5 > 0$, the parabola opens upward.
- The vertex of the parabola occurs at:

$$
x_{vertex} = -\frac{b}{2a} = -\frac{4}{2 \times 5} = -\frac{4}{10} = -0.4
$$

- The y-coordinate of the vertex:

$$
Y_{vertex} = 5(-0.4)^2 + 4(-0.4) = 5(0.16) - 1.6 = 0.8 - 1.6 = -0.8
$$

- Thus, the vertex is at $(-0.4, -0.8)$.

Relationship to the Line Y = 4x

- The parabola touches the line Y = 4x only at the origin, where both curves intersect and are tangent.
- For x > 0, the parabola rises faster than the line, indicating it opens upward.
- For x < 0, the parabola dips below the line, but since they are only tangent at the origin, they do not intersect elsewhere.

Applications and Broader Context

Understanding the specific quadratic curve described has practical applications:

- Engineering Design: Designing parabolic reflectors or antennas where tangency conditions optimize signal focus.
- Physics: Modeling projectile trajectories with specific constraints.
- Mathematics Education: Demonstrating the process of solving systems of equations and understanding tangency.
- Computer Graphics: Rendering curves with precise tangency points for smooth transitions.

Summary of Key Findings

- The quadratic curve passing through (2, 28) and tangent to Y = 4x at the origin is:

$$
Y = 5x^2 + 4x
$$

- It satisfies all conditions: passing through the point, passing through the origin, and having matching slopes at the tangent point.

- The vertex of this parabola is at $(-0.4, -0.8)$, indicating its minimum point, and it opens upward.

- The tangency condition ensures the parabola and the line touch only at the origin, making this a unique geometric configuration.

Conclusion

This comprehensive analysis demonstrates how to determine a quadratic equation based on geometric constraints involving points and tangents. The approach combines algebraic methods with calculus, illustrating the interconnectedness of different mathematical concepts. Whether for academic purposes, engineering applications, or computational modeling, understanding these relationships enhances problem-solving skills and deepens comprehension of quadratic functions.

Further Reading and Resources

- "Algebra and Calculus: An Introduction" - for foundational concepts.
- "Analytic Geometry" - for understanding curves, tangency, and geometric properties.
- Online graphing calculators - to visualize quadratic curves and tangency points.
- Educational videos on derivatives and quadratic equations - for visual learners.

Keywords: quadratic curve, parabola, tangent line, point passing through, coefficients, derivatives, geometric interpretation, quadratic function, $y = ax^2 + bx + c$, tangent at origin, vertex, applications of quadratic equations

Frequently Asked Questions

What is the general form of the quadratic curve discussed in the problem?

The quadratic curve is given by the equation $y = ax^2 + bx + c$.

What specific point does the curve pass through according to the problem?

The curve passes through the point (2, 28).

What is the line to which the curve is tangent at the origin?

The curve is tangent to the line $y = 4x$ at the origin.

How can the point (2, 28) be used to find the coefficients a, b, and c?

By substituting $x=2$ and $y=28$ into the quadratic equation, we obtain an equation relating a, b, and c: $28 = 4a + 2b + c$.

What condition must the quadratic satisfy at the origin to be tangent to y = 4x?

At $x=0$, the quadratic must have a point $(0, c)$ on the curve, and its slope (derivative) at $x=0$ must equal 4, so $y' = 2ax + b = 4$ when $x=0$, giving $b=4$.

How do you determine the value of c using the tangent condition?

Since the curve passes through the origin, substituting $x=0$ gives $y=c$. Because the line passes through $(0,0)$, c must be 0 for the curve to pass through the origin, but considering the point $(2,28)$ helps refine the coefficients.

What is the derivative of the quadratic function y = ax^2 + bx + c?

The derivative is $y' = 2ax + b$.

How do the tangent line's slope and the derivative relate at the point of tangency?

At the point of tangency (the origin), the derivative y' must equal the slope of the line $y=4x$, which is 4.

What steps are involved in solving for the coefficients a, b, and c?

First, use the point $(2,28)$ to get an equation, then set y' at $x=0$ equal to 4 to find b, assume $c=0$ if passing through origin, and solve the resulting equations for a, b, and c.

What are the final values of a, b, and c for the quadratic passing through (2,28) and tangent to y=4x at the origin?

The coefficients are $a = 3$, $b = 4$, and $c = 0$, resulting in the quadratic $y = 3x^2 + 4x$.

Additional Resources

Comprehensive Analysis of the Quadratic Curve $Y = ax^2 + bx + c$: Passing Through $(2,28)$ and Tangent to $Y=4x$ at the Origin

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Introduction

Mathematics often presents intriguing problems that challenge our understanding of geometric and algebraic relationships. One such problem involves analyzing a quadratic curve defined by the equation $Y = ax^2 + bx + c$. The specific conditions given are that this parabola passes through the point $(2,28)$ and is tangent to the line $Y=4x$ at the origin $(0,0)$. This problem encapsulates key concepts in quadratic functions, tangency, and geometric interpretation, offering a rich exploration into the properties of conic sections and their intersections with lines.

This detailed review aims to systematically dissect these conditions, derive the necessary parameters a, b, and c, and explore the implications of the given constraints. We will delve into the algebraic derivations, analyze the geometric relationships, and examine potential applications and extensions of this problem.

Fundamental Concepts and Definitions

Quadratic Functions and Parabolas

A quadratic function is expressed as:

$$
Y = ax^2 + bx + c
$$

where $a \neq 0$. Its graph is a parabola, which can open upwards ($a > 0$) or downwards ($a < 0$). The parameters b and c influence the shape, position, and orientation of the parabola.

Tangency and Contact Point

A line is tangent to a curve at a point if they touch without crossing, sharing a common point and having the same slope at that point. Mathematically, for a parabola $Y = f(x)$ and a line $Y = L(x)$, the tangent point x_t satisfies:

$$
f(x_t) = L(x_t)
$$
and
$$
f'(x_t) = L'(x_t)
$$

where $f'(x)$ and $L'(x)$ are the derivatives, representing the slopes at x_t.

The Line $Y=4x$ and its Properties

The line $Y=4x$ has a constant slope of 4 everywhere. At the origin $(0,0)$, it passes through the point with zero intercept and has a slope of 4, which will be crucial in establishing the tangency condition with the parabola.

Step-by-Step Analysis

1. Establishing the Conditions

Given the problem, the parabola:

$$
Y = ax^2 + bx + c
$$

must satisfy:

- Passes through $(2, 28)$:

$$
28 = a(2)^2 + b(2) + c = 4a + 2b + c
$$

- Is tangent to the line $Y=4x$ at the origin $(0,0)$:

- The parabola passes through $(0,0)$:

$$
0 = a(0)^2 + b(0) + c \Rightarrow c = 0
$$

- The slopes are equal at the point of tangency:

$$
f'(x) = 2ax + b
$$

At $x=0$:

$$
f'(0) = 2a(0) + b = b
$$

Since the line has slope 4, and they're tangent at the origin:

$$
f'(0) = 4 \Rightarrow b = 4
$$

Thus, the parameters b and c are immediately determined:

$$
b = 4, \quad c = 0
$$

2. Deriving the Parameter a

With $b=4$ and $c=0$, substitute into the passing-through point condition:

$$
28 = 4a + 2(4) + 0 \Rightarrow 28 = 4a + 8
$$

$$
4a = 20 \Rightarrow a = 5
$$

Result:

$$
a = 5, \quad b = 4, \quad c = 0
$$

The quadratic function thus becomes:

$$
Y = 5x^2 + 4x
$$

Geometric and Analytical Implications

1. Equation of the Parabola

The specific parabola satisfying the conditions is:

$$
Y = 5x^2 + 4x
$$

This parabola opens upwards (since $a=5 > 0$) and passes through the origin with a tangent line $Y=4x$. At the origin, the parabola and line touch smoothly, sharing both the point and the slope.

2. Confirming the Point $(2,28)$

Verify that the parabola passes through $(2, 28)$:

$$
Y = 5(2)^2 + 4(2) = 5 \times 4 + 8 = 20 + 8 = 28
$$

Confirmed.

3. Tangency at the Origin

- The parabola at $x=0$:

$$
Y(0) = 0
$$

- The slope at $x=0$:

$$
f'(x) = 10x + 4
$$
$$
f'(0) = 4
$$

which matches the slope of $Y=4x$.

- The point of contact:

$$
(0,0)
$$

- The line $Y=4x$ at $x=0$:

$$
Y=0
$$

which confirms the tangency.

Deeper Insights and Extensions

1. Visualization of the Parabola and Line

Graphically, the parabola:

- Passes through $(0,0)$ with a tangent line $Y=4x$,
- Passes through $(2,28)$,
- Opens upwards.

The tangent point at the origin is a point of smooth contact, with the parabola just "touching" the line without crossing it locally.

2. Behavior Near the Origin

- Slope comparison:

At $x=0$, both the parabola and the line have slope 4.

- Curvature:

Since $a=5$, the parabola has a relatively steep upward curvature, which causes it to diverge from the tangent line as x increases or decreases from 0.

3. Intersection with the Line $Y=4x$

Beyond the tangent point, the parabola and the line intersect at other points if they cross. To find all intersection points:

$$
5x^2 + 4x = 4x
$$

\[

$$
5x^2 + 4x - 4x = 0
$$

$$
5x^2 = 0
$$

$$
x=0
$$

This indicates that the only intersection point between the parabola and the line is at $x=0$. The tangent point is the unique intersection, emphasizing the nature of tangency.

Broader Mathematical Context and Applications

1. Significance in Calculus and Geometry

This problem exemplifies the concept of tangency and derivative analysis:

- Derivative conditions determine the slope at the point of contact.
- Point conditions specify where the curve passes.

Such problems are foundational in calculus, especially in understanding tangent lines, optimization, and curve fitting.

2. Applications in Engineering and Physics

- Designing curves that are tangent to specific lines is essential in mechanical design, computer graphics, and trajectory planning.
- Ensuring smooth transitions, such as in road or track design, relies on conditions similar to tangency.

3. Extension to Other Conditions

- Multiple tangency points: Adjusting parameters to create multiple tangent points.
- Higher-order polynomials: Extending concepts to cubic or quartic functions.
- Dynamic systems: Analyzing how curves evolve under parameter changes.

Summary and Final Remarks

This detailed exploration of the quadratic curve $Y=ax^2 + bx + c$, passing through $(2,28)$ and tangent to $Y=4x$ at the origin, reveals the precise parameters:

$$
a=5, \quad b=4, \quad c=0
$$

and the resulting equation:

$$
\boxed{
Y = 5x^2 + 4x
}
$$

This parabola exhibits a perfect tangent relationship with the line at the origin, with the point and slope conditions satisfied seamlessly. The problem underscores key principles of algebra and calculus, illustrating how geometric intuition complements algebraic derivations.

In conclusion, such problems exemplify the elegant interplay between algebraic equations and geometric interpretations, serving as powerful tools in mathematical analysis, problem-solving, and practical applications across various scientific and engineering disciplines.

Further Exploration Recommendations:

- Graph the parabola and line to visualize the tangent contact.
- Investigate how changing the point \((2,28)\) affects the parabola.
- Explore similar problems with different lines or points.
- Study the behavior of the parabola near the tangent point to understand curvature effects.

This comprehensive analysis not only solves the given problem but also provides a deep understanding of the underlying mathematical principles, enriching one's appreciation of

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