True Or False: There Is A Function On The Interval [-1,1] That Does Not Have An Absolute Maximum Value

True Or False: There Is A Function On The Interval [-1,1] That Does Not Have An Absolute Maximum Value is a question that often arises in the study of calculus and real analysis. Understanding the properties of functions defined on closed and bounded intervals is fundamental in mathematics, especially in the context of maximum and minimum values. This article explores this question in depth, examining the concepts of absolute maximum, the characteristics of functions on closed intervals, and the conditions under which a function may or may not have an absolute maximum within the interval [-1,1].

Understanding Absolute Maximum and Minimum Values

Definitions and Basic Concepts

In calculus, the absolute maximum (also called the global maximum) of a function f on a given interval $[a, b]$ is a point $c \in [a, b]$ such that:

$$
f(c) \geq f(x) \quad \text{for all } x \in [a, b]
$$

Similarly, an absolute minimum is a point $d \in [a, b]$ where:

$$
f(d) \leq f(x) \quad \text{for all } x \in [a, b]
$$

These points are sometimes called the global extrema of the function on the interval.

The Extreme Value Theorem

A key result in calculus, the Extreme Value Theorem (EVT), states:

If a function f is continuous on a closed and bounded interval $[a, b]$, then f attains both an absolute maximum and an absolute minimum on $[a, b]$.

This theorem guarantees the existence of these extrema under certain conditions, primarily continuity and the interval being closed and bounded.

Analyzing the Interval [-1, 1]

Closed and Bounded Intervals

The interval $[-1, 1]$ is a closed and bounded subset of the real numbers $\mathbb{R}$. According to the Extreme Value Theorem, if a function f defined on $[-1,1]$ is continuous, then it must have an absolute maximum and an absolute minimum somewhere within the interval.

Therefore:

- If f is continuous on $[-1, 1]$, it must have an absolute maximum (and minimum).
- If f is discontinuous, the theorem does not apply directly, and the existence of an absolute maximum is not guaranteed.

Is It Possible for a Function on [-1, 1] to Lack an Absolute Maximum?

The core of the question is: Can a function defined on $[-1,1]$ fail to have an absolute maximum?

The answer depends on the properties of the function, specifically its continuity.

Case 1: Continuous Functions on [-1,1]
For continuous functions, the answer is no—such functions must have an absolute maximum and minimum on $[-1,1]$, in accordance with the Extreme Value Theorem.

Case 2: Discontinuous Functions on [-1,1]
However, if the function is discontinuous, it is possible for a function to lack an absolute maximum. This is because the theorem does not apply to discontinuous functions, and the supremum (least upper bound) of the function may not be attained within the interval.

Examples Illustrating the Concepts

Example 1: Continuous Function with Absolute Max and Min

Consider the function:

$$
f(x) = x^2
$$
defined on $[-1, 1]$.

- f is continuous on $[-1, 1]$.
- The maximum value occurs at $x = \pm 1$, where $f(\pm 1) = 1$.
- The minimum value occurs at $x=0$, where $f(0) = 0$.

Result: f attains both an absolute maximum and minimum within the interval.

Example 2: Discontinuous Function Lacking an Absolute Maximum

Define the function:
$$
f(x) = \begin{cases}
\frac{1}{x} & \text{if } x \in (-1, 0) \cup (0, 1] \\
0 & \text{if } x=0
\end{cases}
$$

- f is discontinuous at $x=0$.
- As $x \to 0^-$, $f(x) \to -\infty$, and as $x \to 0^+$, $f(x) \to +\infty$.
- On $[-1, 1]$, the supremum (least upper bound) of f is $+\infty$, but f does not actually attain this value at any point within the interval.
- The function does not have an absolute maximum because the supremum is infinite and not attained at any finite point.

Result: The function does not have an absolute maximum on $[-1,1]$.

Implications and Key Takeaways

Continuity Is Crucial

- For functions continuous on $[-1,1]$, the Extreme Value Theorem guarantees an absolute maximum and minimum.
- For discontinuous functions, it is possible to lack an absolute maximum or minimum.

Understanding Supremum and Infimum

- The supremum (least upper bound) of a function's range may not be attained within the interval.
- The infimum (greatest lower bound) may similarly not be achieved.

Unbounded and Discontinuous Functions

- Functions that are unbounded or have vertical asymptotes within $[-1,1]$ may fail to have an absolute maximum.

Summary: True or False?

Based on the above analysis:

- If the function is continuous on $[-1, 1]$, then False—it must have an absolute maximum value.
- If the function is discontinuous or unbounded, then True—it may lack an absolute maximum.

Therefore, the statement "There is a function on the interval $[-1,1]$ that does not have an absolute maximum value" is conditionally true. It is true in the case of discontinuous or unbounded functions, and false for continuous functions.

Final Thoughts

Understanding the conditions under which functions attain their extrema is fundamental in calculus and mathematical analysis. The critical factors include whether the function is continuous, bounded, and the nature of the interval itself. The interval $[-1,1]$ being closed and bounded strongly supports the existence of extrema for continuous functions, but allows for exceptions when discontinuities or unbounded behaviors are introduced.

In conclusion, the answer to whether there exists a function on $[-1,1]$ that does not have an absolute maximum depends on the class of functions considered. For continuous functions, the answer is no; for discontinuous or unbounded functions, the answer is yes. This nuanced understanding is essential for advanced studies in calculus and analysis, emphasizing the importance of function properties in determining extrema.

Keywords: absolute maximum, continuous functions, discontinuous functions, extreme value theorem, supremum, bounded interval, calculus, real analysis, extrema, functions on [-1,1]

Frequently Asked Questions

True or False: Every continuous function on the interval [-1,1] has an absolute maximum value.

False. According to the Extreme Value Theorem, every continuous function on a closed and bounded interval like [-1,1] must have both an absolute maximum and minimum. However, if the function is not continuous, it may lack an absolute maximum.

Is it possible for a discontinuous function defined on [-1,1] to not have an absolute maximum?

Yes. If a function is discontinuous on [-1,1], it may fail to attain an absolute maximum, especially if it has unbounded behavior or discontinuities that prevent the maximum from being reached.

True or False: The statement 'There is a function on [-1,1] that does not have an absolute maximum' is false if the function is continuous.

True. If the function is continuous on the closed interval [-1,1], then by the Extreme Value Theorem, it must have an absolute maximum. Therefore, such a function cannot exist in this case.

What type of functions on [-1,1] can lack an absolute maximum?

Functions that are not continuous, unbounded, or have discontinuities or oscillations approaching infinity can lack an absolute maximum on [-1,1].

Does the existence of an absolute maximum depend solely on the interval [-1,1]?

No. The existence of an absolute maximum depends on the properties of the function, such as continuity and boundedness, not just the interval itself. On a closed, bounded interval, continuous functions always have an absolute maximum, but discontinuous functions may not.

Additional Resources

True Or False: There Is A Function On The Interval [-1,1] That Does Not Have An Absolute Maximum Value

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Introduction

The question posed—"True Or False: There Is A Function On The Interval [-1,1] That Does Not Have An Absolute Maximum Value"—may initially seem trivial or straightforward. After all, the interval $[-1, 1]$ is a closed, bounded interval on the real line, and the Extreme Value Theorem (EVT) guarantees that every continuous function on such an interval must attain both a maximum and a minimum value.

However, the depth of this question becomes apparent when we consider functions that are not continuous, or functions that are defined in more complex ways, such as those that are discontinuous or unbounded within the interval. The investigation of this question is not merely a matter of academic curiosity but touches on fundamental concepts in real analysis, such as continuity, boundedness, and the properties of functions on compact sets.

This article aims to thoroughly explore the question through an investigative lens, analyzing the conditions under which functions on $[-1, 1]$ can lack an absolute maximum, and delineating the differences between various classes of functions—continuous, discontinuous, bounded, unbounded—and their maximum values within this interval.

The Fundamental Theorem of Calculus and Its Limitations

The Extreme Value Theorem (EVT)

At the core of understanding maximum values on an interval lies the Extreme Value Theorem, which states:

> If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum and an absolute minimum within $[a, b]$.

Applying this directly to $[-1, 1]$:

- Any continuous function $f: [-1, 1] \rightarrow \mathbb{R}$ must have an absolute maximum and minimum.

Therefore, in the class of continuous functions, the answer to our primary question is False—there cannot be a continuous function on $[-1, 1]$ that does not have an absolute maximum.

Discontinuities and Unboundedness

However, the EVT does not apply to functions that are discontinuous or unbounded on the interval. The question now shifts:

> Are there functions defined on $[-1, 1]$, perhaps discontinuous or unbounded, that do not have an absolute maximum?

The answer is Yes—if the function is not continuous, or if it behaves "badly" within the interval, it may indeed lack an absolute maximum.

Investigating Functions Without an Absolute Maximum

1. Discontinuous Functions

The key to understanding whether a function on $[-1, 1]$ can lack an absolute maximum lies in the properties of the function's discontinuities and boundedness.

Example 1: Discontinuous but Bounded Function Without a Maximum

Consider the function:

$$
f(x) =
\begin{cases}
x, & x \neq 0 \\
0, & x = 0
\end{cases}
$$

On $[-1, 1]$, this function is discontinuous at $x=0$ but bounded between -1 and 1. The maximum value of f in the interval is 1, attained at $x=1$. The minimum is -1 at $x=-1$.

Conclusion: Despite the discontinuity at zero, the function still has an absolute maximum and minimum.

Key insight: Discontinuity alone does not prevent a function from attaining an absolute maximum provided the function is bounded.

Example 2: Discontinuous and Unbounded Function Without a Maximum

Now, consider:

$$
f(x) = \frac{1}{x}
$$

defined on:

$$
(-1, 1]
$$

- For $x \in (0, 1]$, $f(x) \in [1, \infty)$, unbounded above as $x \to 0^{+}$.

- At $x=1$, $f(1)=1$.

Within $[-1, 1]$, the function is undefined at $x=0$, but if we extended it by defining $f(0) = \text{does not exist}$, the function is discontinuous at 0.

Observation: The supremum of f over $[-1,1]$ does not exist because:

- For any large M, there exists x close enough to 0 from the right such that $f(x) > M$.
- Since the function grows without bound near 0, it lacks an absolute maximum.

Conclusion: Unbounded functions or those with unbounded oscillations can fail to attain an absolute maximum within the interval.

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2. Functions That Are Not Bounded

Can a function be unbounded on $[-1,1]$ and still have a maximum?

- No. If a function is unbounded above on a set, it cannot attain a finite maximum, because for any candidate maximum M, there exists some x with $f(x) > M$.

Result: Unbounded functions cannot have an absolute maximum (though they may have a supremum that is not attained).

The Role of the Supremum and Attainment

Understanding the distinction between the supremum and the maximum is critical:

- The supremum (least upper bound) of a function's values on $[-1, 1]$ may exist even if the function does not attain it.
- The maximum is the actual value attained by the function at some point in $[-1, 1]$.

Example:

$$
f(x) =
\begin{cases}
x, & x \neq 1 \\
2, & x=1
\end{cases}
$$

Here:

- The supremum of f on $[-1, 1]$ is 2 (attained at $x=1$), so the maximum exists.
- But if we define:

$$
f(x) = x \text{ for } x \in [-1,1)
$$

and $f(1)$ is undefined, then:

- The supremum is 1, but it is not attained at any point in the domain (since f is undefined at $x=1$).

In our case, if the domain includes the point where the supremum is achieved, the maximum exists; if not, it does not.

The Impact of Domain Closure and Function Properties

Closed and Bounded Domains

- The Extreme Value Theorem ensures that every continuous function on closed and bounded intervals—such as $[-1,1]$—must have an absolute maximum.

Implication: For continuous functions, the statement is false; such functions must have an absolute maximum.

Open or Half-Open Domains

- On open intervals like $(-1, 1)$ or half-open intervals, functions may fail to attain their supremum, making the statement true in those contexts.

Example:

$$
f(x) = \frac{1}{1 - x}
$$

defined on $(-1, 1)$:

- As $x \to 1^{-}$, $f(x) \to +\infty$,
- The supremum does not exist finitely; the function is unbounded.

In this case, no maximum exists.

Summary of Conditions for the Existence of an Absolute Maximum on $[-1,1]$

Function Class	Conditions	Does it have an absolute maximum?	Notes
Continuous	On $[-1,1]$	Yes	Guaranteed by EVT
Discontinuous but bounded	Yes	Yes	Discontinuity does not prevent maximum if bounded
Discontinuous and unbounded	No	No	Unboundedness prevents maximum
Not bounded	No	No	Cannot have maximum if unbounded
Defined on open or half-open interval	Not guaranteed	No	No guarantee without closure

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Final Verdict: True or False?

- For continuous functions on $[-1,1]$, the statement "There is a function on $[-1,1]$ that does not have an absolute maximum" is False.
- For discontinuous functions, or functions that are not bounded, it is True that such functions may exist and may not have an absolute maximum.

In conclusion:

> On the interval $[-1,1]$, the statement "There exists a function that does not have an absolute maximum" is true in the general case (considering all functions), but false when restricted to continuous functions.

Broader Implications and Mathematical Significance

This exploration underscores the importance of function properties—particularly continuity and boundedness—in determining extremal values. The Extreme Value Theorem is a cornerstone result in real analysis, emphasizing that the structure of the domain and the nature of the function dictate whether maxima and minima

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