(1) Show That The Equation X3 X 1 = 0 Has The Unique Solution In [1 2]. (2) Find A Suitable Fixed-point

(1) Show That The Equation X^3 - X - 1 = 0 Has The Unique Solution In [1, 2]. (2) Find A Suitable Fixed-point

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Introduction

Solving equations within a specific interval is a fundamental aspect of mathematical analysis, especially in the context of root-finding and fixed-point theorems. The equation $X^3 - X - 1 = 0$ is a cubic polynomial that has garnered significant interest due to its intriguing properties and applications. This article aims to demonstrate that this equation has a unique solution within the interval [1, 2], and subsequently, to identify a suitable fixed-point related to this equation.

Part 1: Showing the Existence and Uniqueness of the Solution in [1, 2]

Understanding the Equation

The given equation:

$$X^3 - X - 1 = 0$$

is a cubic polynomial function, which we denote as:

$$f(x) = x^3 - x - 1$$

Our goal is to show that there exists exactly one solution $x \in [1, 2]$ such that $f(x) = 0$.

Applying the Intermediate Value Theorem (Existence)

The Intermediate Value Theorem (IVT) states that if a function f is continuous on a closed interval $[a, b]$, and if $f(a)$ and $f(b)$ have opposite signs, then there exists at least one $c \in (a, b)$ such that $f(c) = 0$.

Since $f(x) = x^3 - x - 1$ is a polynomial, it is continuous everywhere, including on [1, 2].

Let's evaluate $f(x)$ at the endpoints:

- At $x=1$:

$$
f(1) = 1^3 - 1 - 1 = 1 - 1 - 1 = -1
$$

- At $x=2$:

$$
f(2) = 2^3 - 2 - 1 = 8 - 2 - 1 = 5
$$

Since $f(1) = -1 < 0$ and $f(2) = 5 > 0$, the function changes sign over the interval [1, 2].

Conclusion: By IVT, there exists at least one $c \in (1, 2)$ such that $f(c) = 0$.

Establishing Uniqueness (Using Monotonicity)

To demonstrate that this solution is unique in [1, 2], we need to analyze the behavior of $f(x)$.

Calculate the derivative:

$$
f'(x) = 3x^2 - 1
$$

- For $x \in [1, 2]$:

$$
f'(x) \geq 3(1)^2 - 1 = 3 - 1 = 2 > 0
$$

and

$$
f'(x) \leq 3(2)^2 - 1 = 3(4) - 1 = 12 - 1 = 11 > 0
$$

Thus, $f'(x) > 0$ for all $x \in [1, 2]$.

Implication:

Since $f'(x) > 0$ on the interval, $f(x)$ is strictly increasing over [1, 2].

Consequently:

- The function crosses zero exactly once in this interval.

Summary:

- The function $f(x)$ is continuous and strictly increasing on [1, 2].
- It changes sign from negative at $x=1$ to positive at $x=2$.

Therefore:

> The equation $x^3 - x - 1 = 0$ has a unique solution in $[1, 2]$.

Part 2: Finding a Suitable Fixed-point

Understanding Fixed-points in Context

A fixed-point of a function $g(x)$ is a point $x^$ such that:

$$
g(x^) = x^
$$

In the context of solving $f(x) = 0$, one common approach is to rewrite the equation in a form suitable for fixed-point iteration.

Given:

$$
x^3 - x - 1 = 0
$$

which can be rearranged in various ways to facilitate fixed-point iteration.

Rearrangement for Fixed-point Iteration

One common approach is to express x as a function of itself:

$$
x = \sqrt[3]{x + 1}
$$

or equivalently,

$$
x = \left(x + 1 \right)^{1/3}
$$

Define:

$$
g(x) = \left(x + 1 \right)^{1/3}
$$

Our goal is to find an x such that:

$$
x = g(x)
$$

which is a fixed point of g.

Checking the Suitability of $g(x) = (x+1)^{1/3}$

To verify whether $g(x)$ is suitable for fixed-point iteration, we analyze:

1. The interval of interest: [1, 2].
2. Whether g maps [1, 2] into itself.
3. The convergence criteria—specifically, whether $|g'(x)| < 1$ in [1, 2].

Mapping [1, 2] Through $g(x) = (x + 1)^{1/3}$

Calculate $g(1)$:

$$
g(1) = (1 + 1)^{1/3} = 2^{1/3} \approx 1.26
$$

Calculate $g(2)$:

$$
g(2) = (2 + 1)^{1/3} = 3^{1/3} \approx 1.44
$$

Since $g(1) \approx 1.26$ and $g(2) \approx 1.44$, the image of [1, 2] under g lies roughly within [1.26, 1.44], which is a subset of [1, 2].

Conclusion: $g(x)$ maps [1, 2] into a subset of itself, making it a candidate for fixed-point iteration.

Convergence Analysis: Derivative of $g(x)$

Calculate:

$$
g'(x) = \frac{1}{3} (x + 1)^{-2/3}
$$

Evaluate at $x=1$:

$$
g'(1) = \frac{1}{3} (1 + 1)^{-2/3} = \frac{1}{3} \times 2^{-2/3}
$$

Since $2^{2/3} = (2^{1/3})^2 \approx (1.26)^2 \approx 1.59$,

$$
g'(1) \approx \frac{1}{3} \times \frac{1}{1.59} \approx \frac{1}{3} \times 0.63 \approx 0.21
$$

Similarly, at $x=2$:

$$
g'(2) = \frac{1}{3} (3)^{-2/3} \approx \frac{1}{3} \times 3^{-2/3}
$$

Calculate $3^{2/3}$:

$$
3^{2/3} = (3^{1/3})^2 \approx (1.442)^2 \approx 2.08
$$

Thus,

$$
g'(2) \approx \frac{1}{3} \times \frac{1}{2.08} \approx 0.16
$$

Since $|g'(x)| < 1$ for all $x \in [1, 2]$, the fixed-point iteration based on $g(x)$ will converge in this interval.

Summary of the Fixed-point Approach

- The function $g(x) = (x + 1)^{1/3}$ has a fixed point in [1, 2].
- It maps [1, 2] into itself.
- Its derivative's absolute value is less than 1 within the interval, ensuring convergence of fixed-point iteration.

Conclusion

This comprehensive analysis confirms that:

- The equation $x^3 - x - 1 = 0$ has exactly one solution within the interval [1, 2], as proven using the Intermediate Value Theorem and the monotonicity of $f(x)$.
- A suitable fixed-point for iterative methods is $g(x) = (x + 1)^{1/3}$, which satisfies the necessary conditions for convergence, making it an effective choice for numerical solution approaches.

By understanding the behavior of the polynomial and selecting an appropriate fixed-point function,

Frequently Asked Questions

How can I show that the equation X^3 + X - 1 = 0 has a unique solution in the interval [1, 2]?

You can apply the Intermediate Value Theorem to demonstrate the existence of a root in [1, 2], then use the derivative to verify that the function is strictly monotonic, ensuring the solution's uniqueness.

What is the role of the derivative in proving the

uniqueness of the solution for the equation $X^3 + X - 1 = 0$?

The derivative $f'(X) = 3X^2 + 1$ is always positive, indicating that the function is strictly increasing on [1, 2]. This monotonicity guarantees that there is only one solution in this interval.

How do I find a fixed point for the equation $X = g(X)$ related to the original equation?

Rearrange the original equation to express it in the form $X = g(X)$. For example, from $X^3 + X - 1 = 0$, you might write $X = (1 - X)^{1/3}$ or another suitable form, then analyze its fixed points.

What criteria should be used to identify a suitable fixed-point function g(X) for iterative methods?

g(X) should be continuous and ideally a contraction on the interval, meaning $|g'(X)| < 1$, to ensure convergence of fixed-point iterations towards the solution.

Can the Banach Fixed-Point Theorem be applied to find the fixed point in this context?

Yes, if you can define a function g(X) such that it is a contraction mapping on [1, 2], then the Banach Fixed-Point Theorem guarantees the existence and uniqueness of the fixed point, which solves the equation.

What iterative method would be suitable for approximating the fixed point of the equation, and how do I select g(X)?

Methods like fixed-point iteration or Newton-Raphson can be used. When choosing g(X), ensure it is derived from the original equation and satisfies contraction conditions for convergence.

How do I verify that a chosen g(X) leads to convergence when finding the fixed point?

Check that $|g'(X)| < 1$ in the interval of interest. If this condition holds, the iterative process will converge to the fixed point starting from an initial guess within the interval.

Additional Resources

Mathematical Analysis and Fixed-Point Identification: An Expert Examination

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Introduction

In the realm of mathematical analysis, particularly within the scope of equations and fixed-point theory, understanding the behavior and solutions of specific functions is crucial. This article delves into a detailed exploration of the equation $x^3 + x - 1 = 0$, demonstrating its unique solution within the interval $[1, 2]$, and subsequently, identifying an appropriate fixed point associated with this equation. Whether you are a student, researcher, or enthusiast, this comprehensive review aims to clarify the underlying concepts, provide rigorous proof, and suggest practical applications.

Part 1: Showing the Equation $x^3 + x - 1 = 0$ Has a Unique Solution in $[1, 2]$

Understanding the Equation

The equation in question,

$$
x^3 + x - 1 = 0,
$$

is a nonlinear polynomial equation of degree three. Its roots — the solutions for x satisfying the equation — are fundamental in various mathematical contexts, including calculus, algebra, and numerical methods.

The goal is to establish that there exists exactly one solution within the interval $[1, 2]$. To do this, we will invoke the Intermediate Value Theorem and properties of continuous functions, along with derivative analysis for uniqueness.

Step 1: Confirm Continuity and Differentiability

The function

$$
f(x) = x^3 + x - 1
$$

is a polynomial; therefore, it is continuous and differentiable for all real numbers, especially within $[1, 2]$.

Step 2: Evaluate $f(x)$ at the Endpoints

To apply the Intermediate Value Theorem (IVT), evaluate f at the bounds:

$$
f(1) = 1^3 + 1 - 1 = 1 + 1 - 1 = 1,
$$

$$
f(2) = 2^3 + 2 - 1 = 8 + 2 - 1 = 9.
$$

Notice that:

$$
f(1) = 1 > 0,
$$

$$
f(2) = 9 > 0.
$$

Since both are positive, at first glance, it appears that the function does not cross zero in $[1, 2]$. However, to confirm the existence of roots, we need to check at other points within the interval, especially at $x = 0.5$, or better, check the behavior around $x = 0$ to see if the function crosses zero elsewhere or evaluate at other points.

But since the function is positive at both ends, the initial evaluation suggests no root in $[1, 2]$. Let's refine our approach.

Re-evaluating the problem:

Actually, the question likely intends to show the solution in $[0,1]$ or a different interval. But since the original prompt states $[1, 2]$, and the function is positive there, perhaps the root lies elsewhere. Alternatively, perhaps the function's roots are in $[0, 1]$. Let's check at $x=0.5$:

$$
f(0.5) = (0.5)^3 + 0.5 - 1 = 0.125 + 0.5 - 1 = -0.375,
$$

which is negative. Now, at $x=1$:

$$
f(1) = 1 + 1 - 1 = 1 > 0.
$$

Since $f(0.5) < 0$ and $f(1) > 0$, by the IVT, there exists at least one root in $(0.5, 1)$.

Similarly, at $x=1.5$:

$$
f(1.5) = (1.5)^3 + 1.5 - 1 = 3.375 + 1.5 - 1 = 3.875 > 0,
$$

and at $x=1$, as above, $f(1)=1>0$. The function remains positive in $[1, 2]$, so no root exists in $[1, 2]$.

Conclusion of Part 1:

- The solution exists in $[0,1]$, specifically between 0.5 and 1.

- In the original problem, if the interval is $[1, 2]$, then the function does not cross zero there, so the root is not in $[1, 2]$.

Assuming a correction: The intended interval could be $[0,1]$ or $[0.5, 1.5]$.

However, if the question is to demonstrate the existence of a unique solution in some interval where the function crosses zero, the standard procedure involves:

- Showing f is continuous on the interval,

- Computing f at the endpoints,

- Applying the Intermediate Value Theorem to confirm the existence of at least one root,

- Using the derivative to confirm uniqueness.

Part 2: Proving Uniqueness of the Solution

Suppose the root exists in $[a, b]$. To establish uniqueness, we analyze the derivative:

$$
f'(x) = 3x^2 + 1,
$$

which is always positive for all real x, since:

$$
3x^2 + 1 \geq 1 > 0.
$$

This indicates that $f(x)$ is strictly increasing over the entire real line.

Implication:

- A strictly increasing function can cross zero at most once.

- Therefore, the solution in the interval where the function crosses zero is unique.

Summary of Part 1

- The function $f(x) = x^3 + x - 1$ is continuous and differentiable everywhere.

- $f'(x) = 3x^2 + 1 > 0$ for all x.

- Since $f(x)$ is strictly increasing, it can have at most one real root.

- By evaluating $f(0.5) = -0.375 < 0$ and $f(1) = 1 > 0$, the root exists uniquely in $(0.5, 1)$.

Hence, the equation $x^3 + x -1=0$ has a unique solution in the interval $[0.5, 1]$.

Part 2: Finding a Suitable Fixed-Point

Understanding Fixed-Points

In the context of functions, a fixed point of a function $g(x)$ is a point $x^{}$ such that:

\[
g(x^) = x^.
\]

Fixed-point theory is foundational in mathematics, underpinning methods like iteration schemes for solving equations—think of the Banach Fixed Point Theorem, which guarantees the existence and uniqueness of fixed points under specific conditions.

To relate the fixed point to our original equation, consider rewriting \(f(x)=0\) in a form suitable for fixed-point iteration.

Constructing a Suitable Fixed-Point Function

Given:

\[
x^3 + x - 1 = 0,
\]

we can rearrange to express \(x\) as a function of itself:

\[
x = 1 - x^3.
\]

Define:

\[
g(x) = 1 - x^3.
\]

Check for fixed points:

\[
x = g(x) = 1 - x^3,
\]

which implies:

\[
x = 1 - x^3,
\]

and solving:

\[

x^3 + x - 1 = 0,
\]

matching our original equation.

Thus, the fixed points of \(g(x) = 1 - x^3\) coincide with the solutions of the original equation.

Evaluating the Suitability of \(g(x) = 1 - x^3\)

For iterative methods, such as fixed-point iteration, the convergence depends on the properties of \(g(x)\), especially:

- The domain of interest,

- The Lipschitz constant \(|g'(x)|\) should be less than 1 in the interval.

Calculate:

\[
g'(x) = -3x^2.
\]

Within an interval where the root is located, say \([a, b]\), choose a suitable interval where \(|g'(x)| < 1\).

Suppose the root is near \(x \in [0.5, 1]\):

\[
|g'(x)| = 3x^2 \leq 3 \times 1^2 = 3,
\]

which is greater than 1, so the iteration may not converge globally.

To improve convergence, consider modifying the fixed-point function, such as:

\[
g(x) = \frac{1 - x}{2},
\]

or other forms,

1 Show That The Equation X3 X 1 0 Has The Unique Solution

In 1 2 2 Find A Suitable Fixed Point

1 Show That The Equation X3 X 1 0 Has The Unique Solution In 1 2 2 Find A Suitable Fixed Point

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