3 For A Decimal Expansion K.d1d2d3d4 , Let (sn) Be Defined As In Discussion 10.3. Prove Sn < K 1 For

Introduction to Decimal Expansions and Sequence Definitions

3 For A Decimal Expansion K.d1d2d3d4 , Let (sn) Be Defined As In Discussion 10.3. Prove Sn < K 1 For is a statement rooted in the analysis of decimal representations of real numbers and the behavior of sequences associated with these expansions. Decimal expansions serve as a foundational concept in real analysis, providing a way to express every real number in terms of an infinite series of digits. The sequence (sn) typically arises from partial sums or approximations related to the decimal expansion of a number K, and understanding its properties is essential for establishing convergence and bounds. This article explores the definitions, the context of the sequence (sn), and provides a rigorous proof that Sn is less than 1 for the given decimal expansion, following the principles laid out in discussion 10.3.

Understanding Decimal Expansions and Associated Sequences

Decimal Expansions of Real Numbers

Any real number K in the interval [0,1) can be represented uniquely as an infinite decimal expansion:

- K = 0.d1d2d3d4 ...

where each digit d_i belongs to the set $\{0, 1, 2, ..., 9\}$. This expansion is constructed iteratively, with each digit representing a successive approximation of K.

Finite Approximations and Partial Sums

To analyze K, mathematicians often consider finite decimal approximations, known as partial sums:

1. For $n \geq 1$, define the partial sum:

   $S_n = d1/10 + d2/10^2 + ... + d_n/10^n$

2. These partial sums approximate K from below (or sometimes from above) depending on the digits and the context.

These sums serve as the foundation for defining sequences (s_n) that approximate K, typically approaching K as n approaches infinity.

Discussion 10.3 and the Definition of the Sequence (sn)

The Context of Discussion 10.3

In the context of discussion 10.3, the sequence (s_n) is constructed based on the decimal expansion of K and is designed to approximate K from below, ensuring that each s_n is less than or equal to K but converges to it as n increases.

Formal Definition of the Sequence (sn)

Given the decimal expansion K = 0.d1d2d3d4 ..., the sequence (s_n) is often defined as:

- s_n = sum_{i=1}^n d_i / 10^i

which is precisely the partial sum S_n, or sometimes with slight modifications to ensure certain properties, such as truncation or adjustment for infinite expansions.

Proving that S_n < K 1 for All n

Statement of the Theorem

Our goal is to prove that for the sequence (s_n) defined as above, the inequality:

- S_n < K + 1

holds for all n. Alternatively, the common form of the statement is that the partial sums are bounded above by 1, i.e., S_n < 1, given that K < 1.

Key Ideas and Strategy for the Proof

The proof relies on the properties of decimal expansions and the way partial sums are constructed. The main points include:

- The decimal expansion of K is less than 1, so d1, d2, ... are digits from 0 to 9, with the first digit d1 possibly zero.

- The partial sums S_n are finite sums that approximate K from below, with the difference between K and S_n decreasing as n increases.

- Recognizing that the maximal value of S_n is achieved when all digits d_i are equal to 9, but since K < 1, the partial sums are always less than 1.

Detailed Proof that Sn < 1 for All n

Step 1: Establishing the Bound for Partial Sums

Recall that:

```
S_n = d1/10 + d2/10^2 + ... + d_n/10^n
```

and each digit d_i is in {0, 1, 2, ..., 9}. Since K = 0.d1d2d3..., and K < 1, it follows that:

- Sum of the digits weighted by powers of 1/10 is less than 1.

- Specifically, the infinite sum:

  ```
  sum_{i=1}^∞ d_i / 10^i = K < 1
  ```

- Thus, for each n, the partial sum S_n satisfies:

$S_n = \sum_{i=1}^n d_i / 10^i < 1$

Step 2: Bounding the Difference Between K and S_n

The difference between K and S_n is:

$K - S_n = \sum_{i=n+1}^\infty d_i / 10^i$

Since each d_i □ 9, the sum after n is bounded above by the geometric series:

$$\sum_{i=n+1}^\infty 9 / 10^i = 9 / 10^{n+1} + 9 / 10^{n+2} + ... = (9 / 10^{n+1}) / (1 - 1/10) = (9 / 10^{n+1}) / (9/10) = 1 / 10^{n}$$

Therefore, the difference between K and S_n is less than 1/10^n.

Step 3: Concluding the Inequality S_n < K + 1

Given that:

- S_n < K + 1
- And since K < 1, it follows that:

$S_n < 1 + K < 2$

but more precisely, because S_n approaches K from below and is always less than 1, the key inequality is:

$S_n < 1$

which confirms that the partial sums are always less than 1, consistent with the fact that K itself is less than 1.

Implications and Significance of the Result

Convergence of the Sequence (sn)

The inequality $S_n < 1$ for all n indicates that the sequence (s_n) is bounded above by 1. Since it is also increasing (when digits are non-negative), it converges to K, which is less than 1. This demonstrates the consistency of decimal representations and their partial sums as approximations of real numbers.

Relation to the Uniqueness of Decimal Expansions

This proof also reinforces the uniqueness of decimal expansions for numbers in [0, 1), except for numbers with two representations (e.g., 0.999... = 1.000...). The partial sums always stay below 1, confirming that the decimal expansion does not exceed the number's true value.

Conclusion

In summary, starting from the decimal expansion $K = 0.d_1d_2d_3...$, the sequence of partial sums (s_n) defined as the sums of the first n digits divided by powers of 10 is always less than 1, provided that $K < 1$. This conclusion follows logically from the properties of decimal digits, the structure of the sums, and the bounds established via geometric series. The proof underscores the fundamental aspects of decimal representations and their convergence, providing a rigorous foundation for understanding how sequences derived from decimal expansions behave relative to their limiting values.

Frequently Asked Questions

What is the main goal of the proof involving the sequence (s_n) and the decimal expansion K.d1d2d3d4?

The main goal is to show that the sequence (s_n), defined as in Discussion 10.3, is always less than the decimal number K, i.e., s_n < K for all n.

How is the sequence (s_n) constructed from the decimal expansion K.d1d2d3d4?

The sequence (s_n) is typically constructed by considering partial sums or truncations of the decimal expansion, often taking the first n digits to form a rational approximation less than or equal to the original number K.

What key property of decimal expansions is used to prove that s_n < K?

The key property is that finite truncations of the decimal expansion always produce approximations less than or equal to the original number K, ensuring that s_n < K for all n, especially when considering the infinite expansion.

Why is it important to establish that s_n < K in the context of discussions about decimal expansions?

Establishing s_n < K helps demonstrate the convergence of the sequence to K from below, which is crucial in understanding the properties of decimal representations and in proofs related to limits and approximations.

What role does Discussion 10.3 play in understanding the inequality $s_n < K$?

Discussion 10.3 provides the definition and properties of the sequence (s_n), including how it approximates K, which is fundamental in proving that each term remains less than K and converges to it.

Can the inequality $s_n < K$ be strict for all n, and under what conditions?

Yes, the inequality can be strict for all n if the decimal expansion of K is non-terminating and non-repeating, ensuring that the partial sums s_n are always strictly less than K, approaching it asymptotically.

Additional Resources

Decimal Expansions and Sequence Convergence: An In-Depth Analysis of S_n and the Limit K

Introduction: Unlocking the Mysteries of Decimal Expansions and Converging Sequences

The realm of real numbers is rich with fascinating properties, especially when it comes to their representations in decimal form. At the heart of understanding these properties lies the analysis of sequences constructed from decimal expansions, which serve as fundamental tools in real analysis. Specifically, sequences built from the partial sums of decimal expansions—denoted here as (S_n)—offer an intriguing pathway to explore convergence, limits, and inequalities.

This article aims to provide a comprehensive, detailed exploration of the statement:

> "For a decimal expansion $K . d_1 d_2 d_3 d_4$, let (S_n) be defined as in Discussion 10.3. Then $S_n < K$ for all n."

We'll dissect this claim, contextualize each component, and rigorously prove the inequality, all while adopting the accessible yet precise tone of a product expert or reviewer. Our objective is to illuminate the intricate dance between decimal expansions, partial sums, and limits, making the content accessible to students, educators, and enthusiasts alike.

Understanding the Building Blocks: Decimal Expansions and Sequences

What is a Decimal Expansion?

A decimal expansion of a real number K between 0 and 1 (or more generally any real number) is an infinite series:

$$
K = 0.d_1 d_2 d_3 d_4 \ldots
$$

where each d_i is a digit from 0 to 9. For example, the number $\pi/4$ can be expressed as:

$$
0.785398\ldots
$$

This expansion can be viewed as an infinite sum:

$$
K = \sum_{i=1}^\infty \frac{d_i}{10^i}
$$

\]

Each partial sum:

$$
S_n = \sum_{i=1}^n \frac{d_i}{10^i}
$$

approximates K from below or above, depending on the digits d_i, and provides a finite decimal approximation.

The Sequence (S_n): Definition and Significance

Defining (S_n)

In the context of Discussion 10.3, the sequence (S_n) is constructed as the sequence of partial sums of the decimal expansion:

$$
S_n = \sum_{i=1}^n \frac{d_i}{10^i}
$$

This sequence captures the decimal expansion's initial n digits as a finite decimal number. As $n \to \infty$, these partial sums tend to the actual number K, which is represented as an infinite series.

The Core Claim: Showing $S_n < K$

The Inequality in Perspective

The statement:

$$
S_n < K \quad \text{for all } n
$$

asserts that each finite partial sum underestimates the actual number K, assuming certain conditions on the digits d_i and the nature of the decimal expansion.

This inequality is fundamental because it formalizes how decimal approximations approach a real number from below, establishing the sequence's monotonic increasing nature and convergence properties.

Delving into the Proof: Why Is $S_n < K$?

1. Understanding the Construction of K and S_n

- The decimal expansion:

$$
K = 0.d_1 d_2 d_3 \ldots
$$

- The partial sum:

$$
S_n = \sum_{i=1}^n \frac{d_i}{10^i}
$$

captures the first n digits exactly.

- The remaining "tail" of the expansion:

$$
R_n = K - S_n = \sum_{i=n+1}^\infty \frac{d_i}{10^i}
$$

represents the difference between the full number K and its n-digit approximation.

2. Analyzing the Tail R_n

- Since each digit d_i is between 0 and 9, the tail:

$$
R_n \leq \sum_{i=n+1}^\infty \frac{9}{10^i}
$$

- Recognize this as a geometric series with ratio $\frac{1}{10}$:

$$
R_n \leq 9 \times \frac{1}{10^{n+1}} \times \frac{1}{1 - \frac{1}{10}} = 9 \times \frac{1}{10^{n+1}} \times \frac{10}{9} = \frac{1}{10^n}
$$

- Therefore:

$$
0 \leq R_n < \frac{1}{10^n}

\]

This inequality indicates that the tail R_n is positive but bounded above by $\frac{1}{10^n}$.

3. Establishing the Inequality $S_n < K$

- Since:

$$
K = S_n + R_n
$$

and

$$
R_n > 0
$$

(because in most decimal expansions, the digits d_{n+1} are non-negative, and the sum of subsequent terms is positive unless the expansion terminates).

- It follows that:

$$
K > S_n
$$

because $R_n > 0$.

Special Note: In cases where the decimal expansion terminates (e.g., 0.4999... vs. 0.5), the inequality still holds because the tail R_n approaches zero, and the finite sum S_n approaches K

from below, with the understanding of the decimal representation's definition (terminating vs. non-terminating).

Formal Proof: The Inequality Holds for All n

Let's formalize the proof:

Assumption: The decimal expansion of K is expressed as:

$$
K = 0.d_1 d_2 d_3 \ldots
$$

with digits $d_i \in \{0,1,2,\ldots,9\}$.

Step 1: Define:

$$
S_n = \sum_{i=1}^n \frac{d_i}{10^i}
$$

and

$$
R_n = K - S_n = \sum_{i=n+1}^\infty \frac{d_i}{10^i}
$$

Step 2: Since each $d_i \geq 0$, the residual R_n is non-negative:

$$
R_n \geq 0
$$

and, more specifically,

$$
R_n > 0
$$

unless the decimal expansion terminates exactly at n digits (which is a special case).

Step 3: The partial sum S_n is always less than K:

$$
S_n = K - R_n < K
$$

because $R_n > 0$.

Step 4: The upper bound for R_n:

$$
R_n \leq \frac{9}{10^{n+1}} + \frac{9}{10^{n+2}} + \ldots = \frac{9}{10^{n+1}} \times \frac{1}{1 - \frac{1}{10}} = \frac{9}{10^{n+1}} \times \frac{10}{9} = \frac{1}{10^n}
$$

which confirms that:

$$
0 < R_n \leq \frac{1}{10^n}
$$

\]

Conclusion: Since \(R_n > 0 \), the partial sum \(S_n \) is strictly less than \(K \):

\[
\boxed{
S_n < K
}
\]

for all \(n \).

Significance and Implications of the Inequality

This inequality is fundamental in understanding the nature of decimal approximations:

- Monotonicity: The sequence \((S_n) \) is strictly increasing because each additional digit improves the approximation.

- Convergence: The sequence converges to \(K \) from below, ensuring that the decimal expansion faithfully captures the number.

- Uniqueness: It reflects the fact that decimal expansions (except for terminating expansions) are unique in representing real numbers, with the partial sums providing ever-closer approximations.

Broader Context: Connection to Real Analysis and Number Theory

The demonstration of $S_n < K$ exemplifies core principles in real analysis concerning limits, sequences, and series. It also connects to number theory topics such as:

- Representation of irrationals: Infinite non-repeating decimal expansions.

- Representation of rationals: Terminating or repeating decimal expansions.

- Approximation of real numbers: Using decimal truncations as rational approximations with known bounds.

Final Remarks: Why This Matters

Understanding that each partial sum S_n is less than the actual number K is not merely a technical detail but a cornerstone of how decimal representations

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