(a) [8 Pts. Find The General Real Solutions Of The System: X= AX, Where A = (3 - 2)4 - 1 (b) [2 Pts.]

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Introduction

Understanding systems of linear equations is fundamental in various fields such as mathematics, engineering, physics, computer science, and economics. One common type of system involves finding the general solution to a matrix equation of the form X = AX, where A is a square matrix, and X is a vector. This form often appears in the study of dynamical systems, population models, and stability analysis.

In this article, we will explore how to determine the general real solutions of the system X = AX, given a specific matrix A. We will analyze the matrix, find its eigenvalues and eigenvectors, and derive the general solution. The problem is divided into two parts: (a) an 8-point question focusing on the main solution process, and (b) a 2-point question, which may involve a specific aspect or a shorter related problem.

Understanding the System X = AX

Before diving into calculations, let's clarify what the system X = AX represents.

- X is a vector in $\mathbb{R}^2$ (since the matrix A appears to be 2x2 based on the context).
- A is a 2x2 matrix.
- The equation X = AX implies that X is an eigenvector of A associated with an eigenvalue of 1, or that the solutions are related to the eigenstructure of A.

The solutions to such a system are closely tied to the eigenvalues and eigenvectors of A. When solving X = AX, the key steps involve:

1. Finding the eigenvalues λ of A.
2. Finding the eigenvectors corresponding to these eigenvalues.
3. Expressing the general solution based on the eigenstructure.

Part (a): Find The General Real Solutions of the System

Step 1: Determine the Matrix A

From the problem statement, the matrix A appears to be given as:

$$
A = \begin{bmatrix}
3 & -2 \\
4 & -1
\end{bmatrix}
$$

This is a standard 2x2 matrix. Our goal is to find all real vectors X satisfying X = AX.

Step 2: Find Eigenvalues of A

To find the eigenvalues, solve the characteristic equation:

$$
\det(A - \lambda I) = 0
$$

where I is the identity matrix.

Calculate:

$$
A - \lambda I =
\begin{bmatrix}
3 - \lambda & -2 \\
4 & -1 - \lambda
\end{bmatrix}
$$

The determinant is:

$$
(3 - \lambda)(-1 - \lambda) - (-2)(4) = 0
$$

Simplify:

$$
(3 - \lambda)(-1 - \lambda) + 8 = 0
$$

Expand:

$$
(3)(-1 - \lambda) - \lambda(-1 - \lambda) + 8 = 0
$$

$$
-3 - 3\lambda + \lambda + \lambda^2 + 8 = 0
$$

Combine like terms:

$$
\lambda^2 - 2\lambda + 5 = 0
$$

Eigenvalues:

$$
\lambda^2 - 2\lambda + 5 = 0
$$

Discriminant:

$$
D = (-2)^2 - 4 \times 1 \times 5 = 4 - 20 = -16
$$

Since $D < 0$, A has complex conjugate eigenvalues:

$$
\lambda_{1,2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i
$$

Step 3: Eigenvalues and Their Nature

The eigenvalues are:

$$
\lambda_1 = 1 + 2i, \quad \lambda_2 = 1 - 2i
$$

Since these are complex conjugates and not real, the matrix A does not have real eigenvalues. Therefore, the general real solutions to $X = AX$ are related to complex eigenvectors, but because the question asks for real solutions, the only real solutions correspond to the eigenvector associated with the eigenvalue $\lambda = 1$. But here, $\lambda = 1$ is not an eigenvalue; the eigenvalues are complex conjugates with real part 1.

Part (a): Find the Real Solutions

Important insight:

- For the system $X = AX$, the solutions are eigenvectors associated with eigenvalues $\lambda = 1$.
- Since A does not have $\lambda = 1$ as an eigenvalue, the only solution to $X = AX$ is the trivial solution:

$$
X = 0
$$

because:

$$
X = AX \implies (A - I)X = 0
$$

and if $\lambda = 1$ is not an eigenvalue, then $A - I$ is invertible, and only the zero vector satisfies the equation.

Confirming:

$$
A - I =
\begin{bmatrix}
3 - 1 & -2 \\
4 & -1 - 1
\end{bmatrix}
=
\begin{bmatrix}
2 & -2 \\
4 & -2
\end{bmatrix}
$$

Calculate its determinant:

$$
(2)(-2) - (-2)(4) = -4 + 8 = 4 \neq 0
$$

Since the determinant is non-zero, $A - I$ is invertible. Therefore:

$$
(A - I)X = 0 \implies X = 0
$$

Conclusion for part (a):

> The only real solution to the system $X = AX$ for the given matrix A is the trivial solution:

$$
\boxed{
X = \mathbf{0}
}
$$

Part (b): Additional Related Question

Given the context, part (b) might involve:

- Finding solutions for different eigenvalues.
- Determining the stability of the system.
- Computing the general solution when eigenvalues are complex.

Suppose the question asks:

> (b) Find the general solution to the differential equation associated with the matrix A.

This involves expressing the solution in terms of eigenvalues and eigenvectors, leading to a solution involving sinusoidal functions due to complex eigenvalues.

General Solution for Systems with Complex Eigenvalues

Step 1: Eigenvalues and Eigenvectors Recap

The eigenvalues are:

$$
\lambda_{1,2} = 1 \pm 2i
$$

The real part ($\mathrm{Re}(\lambda) = 1$) indicates growth or decay, while the imaginary part ($\mathrm{Im}(\lambda) = \pm 2$) indicates oscillation.

Step 2: Find Eigenvectors for $\lambda = 1 + 2i$

Solve:

$$
(A - (1 + 2i)I) \mathbf{v} = 0
$$

Calculate:

$$
A - (1 + 2i)I =
\begin{bmatrix}
3 - (1 + 2i) & -2 \\
4 & -1 - (1 + 2i)
\end{bmatrix}
=
\begin{bmatrix}
2 - 2i & -2 \\
4 & -2 - 2i
\end{bmatrix}
$$

Set up the system:

\[
(2 - 2i) v_1 - 2 v_2 = 0
\]
\[
4 v_1 + (-2 - 2i) v_2 = 0
\]

From the first equation:

\[
(2 - 2i) v_1 = 2 v_2 \implies v_2 = \frac{2 - 2i}{2} v_1 = (1 - i) v_1
\]

Choose \(v_1 = 1\):

\[
v^{(1)} =
\begin{bmatrix}
1 \\
1 - i
\end{bmatrix}
\]

Similarly, the eigenvector for $\lambda = 1 - 2i$ is the complex conjugate:

\[
v^{(2)} =
\begin{bmatrix}
1 \\
1 + i
\end{bmatrix}
\]

Step 3: Write the General Solution

The real solution to the system involves combining these eigenvectors, leading to expressions involving sine and cosine functions.

The general solution for the differential system:

\[
\frac{dX}{dt} = A X
\]

can be written as:

\[
X(t) = e^{\text{Re}(\lambda) t} \left[C_1 \cos(\text{Im}(\lambda) t) \cdot
\text{Re}(\mathbf{v}) - C_2 \sin(\text{Im}(\lambda) t) \cdot \text{Im}(\mathbf{

Frequently Asked Questions

What is the general approach to solving the system X = AX for real solutions?

The general approach involves finding the eigenvalues and eigenvectors of matrix A, then expressing the solutions in terms of these eigenvalues and eigenvectors to determine the general real solutions of the system.

How do you find the eigenvalues of matrix A in the system X = AX where A = [[3, -2], [4, -1]]?

Eigenvalues are found by solving the characteristic equation det(A - λI) = 0. For A = [[3, -2], [4, -1]], compute det([[3 - λ, -2], [4, -1 - λ]]) and solve for λ.

What are the eigenvalues of matrix A = [[3, -2], [4, -1]]?

The eigenvalues are λ = 1 and λ = -1, obtained by solving the characteristic equation λ^2 - 2λ - 1 = 0.

How do eigenvalues influence the form of the general solution to the system X = AX?

Eigenvalues determine the exponential growth or decay in the solutions. For each eigenvalue, the corresponding eigenvector contributes a term to the general solution, often involving exponential functions like e^{λt}.

What is the final form of the general real solutions for the system X = AX with matrix A = [[3, -2], [4, -1]]?

The general real solutions are linear combinations of solutions associated with eigenvalues λ = 1 and λ = -1, involving exponential functions e^{t} and e^{-t} multiplied by their respective eigenvectors, forming a complete set of solutions describing the system's behavior.

Additional Resources

In-Depth Analysis of Finding the General Real Solutions of the System: X = AX, Where A = \(\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}\)

Introduction to Systems of Linear Equations and Matrices

Understanding systems of linear equations is fundamental in linear algebra, with applications spanning engineering, physics, computer science, economics, and beyond. When such systems are expressed in matrix form, they become more manageable and mathematically elegant.

The specific problem in question involves solving the matrix equation:

$$
\mathbf{X} = A\mathbf{X}
$$

where A is a 2x2 real matrix. This form is particularly significant because it relates to the eigenvalues and eigenvectors of A, and understanding the structure of solutions entails a deep dive into eigen-decomposition, matrix diagonalization, and the spectral theorem.

Restating the Problem in Context

Given:

$$
A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}
$$

The task is to find the general real solutions of the system:

$$
\mathbf{X} = A \mathbf{X}
$$

where $\mathbf{X}$ is a vector in $\mathbb{R}^2$.

This is a homogeneous linear system, which can be rewritten as:

$$
(I - A)\mathbf{X} = \mathbf{0}
$$

where I is the identity matrix:

$$
I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

Finding solutions involves identifying the null space of $I - A$, which reduces to an eigenvalue problem.

Fundamental Concepts Involved

Eigenvalues and Eigenvectors

- Eigenvalues (\(\lambda\)) are scalars such that there exists a non-zero vector \(\mathbf{v}\) satisfying:

\[
A\mathbf{v} = \lambda \mathbf{v}
\]

- Eigenvectors (\(\mathbf{v}\)) are the corresponding non-zero vectors associated with these eigenvalues.

- The solutions to \(\mathbf{X} = A \mathbf{X}\) are directly related to eigenvectors corresponding to eigenvalues \(\lambda = 1\), since rearranging yields:

\[
(A - I)\mathbf{X} = \mathbf{0}
\]

and solutions exist only when \(\mathbf{X}\) is an eigenvector of \(A\) with eigenvalue 1.

Step-by-Step Solution Approach

Step 1: Compute \(A - I\)

\[
A - I = \begin{bmatrix} 3 - 1 & -2 \\ 4 & -1 - 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 4 & -2 \end{bmatrix}
\]

The solutions \(\mathbf{X}\) satisfy:

\[
(A - I)\mathbf{X} = \mathbf{0}
\]

This reduces to finding the null space of \(A - I\).

Step 2: Find the Eigenvalues of \(A\)

Alternatively, the eigenvalues of \(A\) satisfy the characteristic equation:

\[
\det(A - \lambda I) = 0
\]

Compute:

\[

A - \lambda I = \begin{bmatrix} 3 - \lambda & -2 \\ 4 & -1 - \lambda \end{bmatrix}
\]

Then,

\[
\det(A - \lambda I) = (3 - \lambda)(-1 - \lambda) - (-2)(4)
\]

Calculating:

\[
(3 - \lambda)(-1 - \lambda) + 8
\]

Expand:

\[
(3)(-1 - \lambda) - \lambda(-1 - \lambda) + 8
\]
\[
-3 - 3\lambda + \lambda + \lambda^2 + 8
\]
\[
(\lambda^2 - 2 \lambda + 5) = 0
\]

which simplifies to the quadratic:

\[
\lambda^2 - 2\lambda + 5 = 0
\]

Solve for \(\lambda\):

\[
\lambda = \frac{2 \pm \sqrt{(-2)^2 - 4 \times 1 \times 5}}{2} = \frac{2 \pm \sqrt{4 - 20}}{2} = \frac{2 \pm \sqrt{-16}}{2}
\]

\[
\lambda = \frac{2 \pm 4i}{2} = 1 \pm 2i
\]

Key observation: Both eigenvalues are complex conjugates and not real. Since the problem asks for real solutions, this indicates the only real solution occurs when the eigenvalue is \(\lambda = 1\).

Step 3: Determine if \(\lambda=1\) is an eigenvalue

From the characteristic polynomial, $\lambda=1$ is not an eigenvalue, because:

$$
\lambda^2 - 2 \lambda + 5 = 0
$$

Substituting $\lambda = 1$:

$$
1 - 2 + 5 = 4 \neq 0
$$

Thus, the eigenvalues are complex, and there is no real eigenvector associated with $\lambda=1$.

Step 4: Implications for the System $\mathbf{X} = A \mathbf{X}$

- Since the eigenvalues are complex conjugates, the only real solution to $\mathbf{X} = A \mathbf{X}$ is the trivial solution:

$$
\boxed{\mathbf{X} = \mathbf{0}}
$$

- No non-trivial real solutions exist, because the only eigenvalue equal to 1 (which would yield non-trivial solutions) does not exist among real eigenvalues.

Summary of Findings

Aspect	Details
Eigenvalues	$\lambda = 1 \pm 2i$ (complex conjugates)
Eigenvectors	Complex, not real
Real solutions	Only $\mathbf{X} = \mathbf{0}$ (trivial solution)
Conclusion	The system $X = A X$ admits only the trivial solution over the reals

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Deep Dive: Theoretical Implications and Broader Context

The Nature of the Solutions

The solutions to the system:

$$
\mathbf{X} = A \mathbf{X}
$$

are precisely the eigenvectors associated with eigenvalues $\lambda = 1$. Since in this case, the eigenvalues are complex, the only real solution is the zero vector. This underscores a fundamental principle: a real matrix with complex eigenvalues does not admit non-trivial real eigenvectors associated with those eigenvalues.

Significance in Matrix Theory

- Diagonalizability: Since the eigenvalues are complex conjugates and the matrix A has real entries, A is diagonalizable over $\mathbb{C}$ but not over $\mathbb{R}$.

- Spectral Theorem: The spectral theorem applies to symmetric matrices; A is not symmetric, so it does not guarantee real eigenvalues or orthogonal eigenvectors.

- Stability Analysis: In systems theory, eigenvalues determine system stability. The presence of eigenvalues with magnitude greater than 1 (here, $\sqrt{1^2 + 2^2} = \sqrt{5} > 1$) indicates instability in discrete systems, with solutions diverging in the complex plane.

Practical Applications and Broader Significance

Understanding the solutions to systems like $\mathbf{X} = A \mathbf{X}$ is crucial in:

- Dynamic Systems Analysis: Eigenvalues dictate long-term behavior; complex eigenvalues lead to oscillatory solutions.

- Quantum Mechanics and Vibrational Analysis: Eigenvalues/eigenvectors represent energy levels or modes.

- Control Systems: Eigenvalues relate to system stability; complex eigenvalues with magnitudes greater than 1 may signal oscillations or divergence.

- Numerical Methods: Eigenvalue computations underpin algorithms for matrix power iteration, PCA, and spectral clustering.

Extending the Analysis: When Do Non-trivial Solutions Exist?

Since the above matrix A does not have eigenvalues equal to 1 over the reals, the only solution is trivial. However, if one considers other matrices where:

- $\lambda = 1$ is an eigenvalue, then non-trivial solutions exist, forming an eigenspace.

- Systems where A is idempotent ($A^2 = A$) lead to solutions where $X = A X$ holds for all vectors in the eigenspace associated with eigenvalue 1.

Final Thoughts and Summary

This detailed analysis reveals that the core of solving the system \(\mathbf{X} = A \mathbf{X}\) hinges on the eigenvalues and eigenvectors of \(A\). For the specific matrix:

\[
A = \begin{bmatrix} 3 & -2 \\ 4

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