Verify That Stokes Theorem Is True For The Vector Field F(x, Y, Z) = Hx, Y, Zi, Where S Is The Part Of

Verify That Stokes Theorem Is True For The Vector Field F(x, Y, Z) = Hx, Y, Zi, Where S Is The Part Of the surface in space, is a fundamental problem in vector calculus that illustrates the profound connection between surface integrals and line integrals. Stokes' Theorem provides a powerful tool for converting a surface integral of the curl of a vector field into a line integral around the boundary of the surface, simplifying complex calculations and enhancing our understanding of vector fields in three-dimensional space. In this article, we will explore the process of verifying the theorem for the specified vector field $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$, considering a given surface S, and demonstrate step-by-step how the theorem holds true through detailed calculations and geometric interpretations.

Understanding Stokes' Theorem

Statement of Stokes' Theorem

Stokes' Theorem states that for a smooth surface S with boundary curve C, and a vector field $\mathbf{F}$, the following equality holds:

$$
\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r}
$$

Where:
- $\nabla \times \mathbf{F}$ is the curl of $\mathbf{F}$,
- $d\mathbf{S}$ is the vector surface element,
- C is the boundary curve of S,
- $d\mathbf{r}$ is the differential element along the curve C.

This theorem links the circulation of the vector field around the boundary C to the flux of its curl across the surface S.

Defining the Vector Field $\mathbf{F}(x, y, z)$

Given the vector field:

$$
\mathbf{F}(x, y, z) = \langle x, y, z \rangle
$$

This is a simple, radial vector field pointing outward from the origin, with components directly proportional to the position coordinates. Its curl, divergence, and other properties are straightforward to compute, making it an ideal candidate for verifying Stokes' Theorem.

Calculating the Curl of $\mathbf{F}$

The curl of $\mathbf{F}$, $\nabla \times \mathbf{F}$, in component form is:

$$
\nabla \times \mathbf{F} = \left|
\begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
x & y & z
\end{array}
\right|
$$

Computing the determinant:

$$
\nabla \times \mathbf{F} = \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z} \right) \mathbf{i} - \left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z} \right) \mathbf{j} + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right) \mathbf{k}
$$

Since all derivatives of variables with respect to different variables are zero, we get:

$$
\nabla \times \mathbf{F} = (0 - 0) \mathbf{i} - (0 - 0) \mathbf{j} + (0 - 0) \mathbf{k} = \mathbf{0}
$$

Thus, the curl of $\mathbf{F}$ is zero everywhere:

$$
\nabla \times \mathbf{F} = \mathbf{0}
$$

This simplifies the verification process significantly because the surface integral of the curl over S must be zero.

Choosing the Surface S and Its Boundary C

To proceed with the verification, we need to specify the surface S and its boundary C. For simplicity, assume S is a standard surface, such as a disk or a portion of a plane, with a well-defined boundary.

Example: S as a Disk in the XY-Plane

Suppose S is the disk of radius R lying in the xy-plane, centered at the origin:

$$
S = \{ (x, y, 0) \mid x^2 + y^2 \leq R^2 \}
$$

The boundary C is the circle:

$$
C = \{ (x, y, 0) \mid x^2 + y^2 = R^2 \}
$$

The orientation of C is counterclockwise when viewed from above (positive z-direction).

Verifying the Surface Integral of $\nabla \times \mathbf{F}$ over S

Since $\nabla \times \mathbf{F} = \mathbf{0}$, the surface integral simplifies:

$$
\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_S \mathbf{0} \cdot d\mathbf{S} = 0
$$

This confirms that the flux of the curl over the surface is zero, regardless of the surface's shape.

Calculating the Line Integral of $\mathbf{F}$ over Boundary C

According to Stokes' Theorem, this line integral should also be zero.

Parameterization of C

The boundary circle C can be parameterized as:

$$
\mathbf{r}(t) = \langle R \cos t, R \sin t, 0 \rangle, \quad t \in [0, 2\pi]
$$

The differential $d\mathbf{r}$ is:

$$
d\mathbf{r} = \langle -R \sin t, R \cos t, 0 \rangle dt
$$

The vector field along C:

$$
\mathbf{F}(\mathbf{r}(t)) = \langle R \cos t, R \sin t, 0 \rangle
$$

Compute the line integral:

$$
\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} dt
$$

Calculating the dot product:

$$
\mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} = \langle R \cos t, R \sin t, 0 \rangle \cdot \langle -R \sin t, R \cos t, 0 \rangle
$$

$$
= R \cos t \times (-R \sin t) + R \sin t \times R \cos t + 0
$$

$$
= - R^2 \cos t \sin t + R^2 \sin t \cos t = 0
$$

Integrating over t:

$$
\int_0^{2\pi} 0 \, dt = 0
$$

This confirms that the line integral around C is zero.

Conclusion: Verifying Stokes' Theorem

Since both the surface integral of the curl and the line integral of the vector field over the boundary are zero, we have verified that Stokes' Theorem holds for the specified vector field $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$ over the chosen surface S.

Summary of key points:

- The curl of $\mathbf{F}$ is zero everywhere.
- The surface integral of the curl over any surface S is therefore zero.
- The line integral of $\mathbf{F}$ around the boundary C of S is also zero.
- The equality prescribed by Stokes' Theorem is satisfied, confirming its validity for this case.

Further Considerations and Generalizations

While this example demonstrates a straightforward verification with a simple vector field and surface, more complex scenarios involve non-zero curl fields and surfaces with more intricate boundary curves. In such cases, the process involves:

- Computing the curl of $\mathbf{F}$,
- Choosing an appropriate surface S,
- Parameterizing the boundary C,
- Calculating both the surface and line integrals explicitly,
- Confirming their equality as per Stokes' Theorem.

This rigorous approach is essential in fields such as electromagnetism, fluid dynamics, and differential geometry, where understanding the interplay between circulation and flux is crucial.

Summary and Final

Frequently Asked Questions

What is Stokes' theorem and how does it relate to the vector field F(x, y, z) = (x, y, z)?

Stokes' theorem relates the surface integral of the curl of a vector field over a surface S to the line integral of the vector field over the boundary curve of S. For F(x, y, z) = (x, y, z), it states that $\int_S (\text{curl } F) \cdot dS = \oint_{\partial S} F \cdot dr$, confirming the theorem's validity in this case.

How do you compute the curl of the vector field F(x, y, z) = (x, y, z)?

The curl of F is computed as curl F = $\nabla \times F = (\partial z/\partial y - \partial y/\partial z, \partial x/\partial z - \partial z/\partial x, \partial y/\partial x - \partial x/\partial y)$. For F(x, y, z) = (x, y,

z), this simplifies to (0, 0, 0), since all partial derivatives are straightforward.

What is the specific surface S and boundary curve ∂S in the problem?

The problem states S as a part of a surface, such as a portion of a sphere, plane, or other surface, with a boundary curve ∂S. The exact surface depends on the given description, but typically involves a well-defined, smooth surface with a closed boundary.

How do you verify Stokes' theorem for the given vector field and surface?

To verify, calculate the surface integral of curl F over S and the line integral of F over ∂S, then check if both are equal. For $F(x, y, z) = (x, y, z)$, since curl $F = 0$, both integrals should be zero, confirming the theorem.

What are the key steps in performing the surface integral of curl F over S?

First, parametrize the surface S, compute the curl of F (which is zero in this case), determine the surface element dS, and evaluate the integral $\int_S (\text{curl } F) \cdot dS$. Since curl $F = 0$, the integral typically evaluates to zero.

How does the boundary curve ∂S influence the line integral in verifying Stokes' theorem?

The boundary curve ∂S is used to compute the line integral $\oint_{\partial S} F \cdot dr$. Its shape and parametrization are essential; for example, if ∂S is a circle, the integral can be computed directly using parametrization of that circle.

What are common challenges faced when verifying Stokes' theorem for a vector field like F(x, y, z)?

Challenges include correctly parametrizing the surface and boundary curve, computing the curl accurately, and ensuring the orientations of the surface and boundary are consistent. Additionally, handling complex surfaces can complicate the integrals.

What conclusions can be drawn if the surface and line integrals are equal for the given vector field?

If both integrals are equal, it confirms that Stokes' theorem holds for the vector field F(x, y, z) = (x, y, z) over the chosen surface S and its boundary ∂S, validating the theorem in this instance.

Why is it important to verify Stokes' theorem for specific

vector fields?

Verifying Stokes' theorem for specific fields helps build understanding of vector calculus concepts, confirms theoretical results with concrete examples, and aids in solving practical problems involving flux and circulation in physics and engineering.

Additional Resources

Verify That Stokes Theorem Is True For The Vector Field F(x, y, z) = (x, y, z), Where S Is The Part Of

Stokes' theorem is a fundamental result in vector calculus that links the surface integral of the curl of a vector field over a surface to the line integral of the vector field itself along the boundary curve of that surface. Verifying the theorem for specific vector fields and surfaces not only deepens our understanding of the theorem's applications but also solidifies its conceptual foundation. In this article, we focus on verifying Stokes' theorem for the vector field F(x, y, z) = (x, y, z), considering a specific surface S. We will methodically analyze the theorem's statement, compute the relevant integrals, and interpret the results to confirm the theorem's validity in this context.

Understanding Stokes' Theorem

Statement of the Theorem

Stokes' theorem states that for a smooth surface S with boundary curve $C = \partial S$, and a vector field $\mathbf{F}$, the following equality holds:

$$
\iint_{S} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{C} \mathbf{F} \cdot d\mathbf{r}
$$

Where:
- $\nabla \times \mathbf{F}$ is the curl of $\mathbf{F}$,
- $d\mathbf{S}$ is the oriented surface element,
- C is the boundary curve of S,
- $d\mathbf{r}$ is the line element along C.

This theorem bridges surface and line integrals, providing a powerful tool for evaluating circulation and flux in various physical and mathematical settings.

Significance and Applications

- Simplifies computation of circulation or flux in complex fields.
- Provides insight into the topology and geometry of vector fields.
- Underpins many physical laws, including electromagnetism and fluid dynamics.

Defining the Vector Field and Surface

Vector Field $\mathbf{F}(x, y, z) = (x, y, z)$

The chosen vector field is a simple, radially symmetric field pointing outward from the origin, with components directly proportional to the position coordinates.

Features of $\mathbf{F}$:
- Smooth and continuously differentiable everywhere.
- Divergence is $\nabla \cdot \mathbf{F} = 3$.
- Curl is $\nabla \times \mathbf{F} = \mathbf{0}$, since the field is irrotational.

Choice of Surface S

For verification, we must specify a surface with a boundary curve \(C \).

Example surface:
- A unit disk in the \(xy \)-plane, centered at the origin, with radius \(R = 1 \).
- The boundary curve \(C \) is the circle \(x^2 + y^2 = 1 \), \(z = 0 \).

This choice simplifies calculations and allows us to explicitly compute both integrals.

Computing the Surface Integral of the Curl of \(\mathbf{F} \)

Calculating \(\nabla \times \mathbf{F} \)

Given \(\mathbf{F} = (x, y, z) \),
\[
\nabla \times \mathbf{F} =
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\partial_x & \partial_y & \partial_z \\
x & y & z
\end{vmatrix}

= (0, 0, 0)
\]
since the partial derivatives of the components are constants, and the determinant simplifies to zero.

Implication:
- The curl of $\mathbf{F}$ is the zero vector field.

Surface Integral of the Curl

Because $\nabla \times \mathbf{F} = \mathbf{0}$,
$$
\iint_{S} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = 0
$$
regardless of the surface S as long as it is smooth and the curl remains zero over S.

Calculating the Line Integral of $\mathbf{F}$ Along C

Parameterization of Boundary Curve C

The boundary circle can be parameterized as:

$$
\mathbf{r}(t) = (\cos t, \sin t, 0), \quad t \in [0, 2\pi]
$$
with differential:
$$
d\mathbf{r} = (-\sin t, \cos t, 0) dt
$$

Evaluating $\mathbf{F}$ Along C

Along C:
$$
\mathbf{F}(\mathbf{r}(t)) = (\cos t, \sin t, 0)
$$

The line integral becomes:
$$
\oint_{C} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} dt
$$
$$
= \int_0^{2\pi} (\cos t, \sin t, 0) \cdot (-\sin t, \cos t, 0) dt
$$
$$
= \int_0^{2\pi} [-\cos t \sin t + \sin t \cos t + 0] dt = \int_0^{2\pi} 0 dt = 0
$$

Comparison and Verification of Stokes' Theorem

Results Summary

- Surface integral of curl: 0
- Line integral around boundary: 0

Since both integrals are equal, Stokes' theorem is verified for the vector field $\mathbf{F}(x, y, z) = (x, y, z)$ over the unit disk in the xy-plane.

Discussion and Generalizations

Key Observations

- The curl of $\mathbf{F} = (x, y, z)$ is zero, simplifying the verification.
- The boundary integral also evaluates to zero due to symmetry and the nature of the vector field.
- The theorem holds true in this specific case,

confirming its validity.

Extensions and Other Surfaces

- For surfaces with non-zero curl, the surface integral would need to be explicitly computed.
- Different surfaces with the same boundary C should yield the same line integral, illustrating the independence of the surface integral on the choice of surface bounded by C.

Potential features and considerations:
- The choice of surface affects the calculation but not the validity of the theorem.
- For more complex vector fields, explicit curl computations become crucial.
- Numerical methods can be employed for complicated surfaces or vector fields.

Pros and Cons of the Verification Approach

Pros:
- Clear demonstration of the theorem's application in a simple case.
- Reinforces understanding of vector calculus concepts

like curl, line, and surface integrals.
- Highlights the importance of parameterization and symmetry.

Cons:
- The example is somewhat trivial due to the zero curl.
- More complex fields require extensive calculations.
- Assumes smoothness and simple surfaces, which may not always be the case.

Conclusion

By explicitly computing both the surface integral of the curl of $\mathbf{F}$ and the line integral along its boundary, we have verified Stokes' theorem for the vector field $\mathbf{F}(x, y, z) = (x, y, z)$ over a unit disk in the xy-plane. The calculations confirm that both integrals are zero, aligning perfectly with the theorem's statement. This exercise exemplifies the power and elegance of Stokes' theorem, illustrating how surface and line integrals are interconnected. Such verifications are essential in understanding the underlying principles of vector calculus and their applications across physics, engineering, and mathematics.

In summary:
- The curl of the field was zero, simplifying the surface integral.
- The boundary integral also evaluated to zero due to symmetry.
- The results confirmed the validity of Stokes' theorem in this context, reinforcing its theoretical and practical significance.

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