Verify That The Function Satisfies The Three Hypotheses Of Rolle's Theorem On The Given Interval. Then

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Understanding the application of Rolle's Theorem is fundamental in calculus, especially in the study of differentiable functions. This theorem provides a significant link between the behavior of a function and its derivatives, serving as a foundational concept that underpins many other theorems and techniques in analysis. In this article, we will explore how to verify that a given function satisfies the three hypotheses of Rolle's Theorem on a specific interval, discuss the significance of each hypothesis, and demonstrate the process with illustrative examples.

What Is Rolle's Theorem?

Rolle's Theorem states that if a function f is continuous on a closed interval [a, b], differentiable on the open interval (a, b), and if the function's values at the endpoints are equal (f(a) = f(b)), then there exists at least one point c in (a, b) where the derivative of the function is zero, i.e., f'(c) = 0.

Formally, the theorem can be expressed as:

> If f is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$, then there exists some $c \in (a, b)$ such that $f'(c) = 0$.

This theorem is essential because it guarantees the existence of a horizontal tangent at some point within the interval, given the specified conditions.

The Three Hypotheses of Rolle's Theorem

To successfully apply Rolle's Theorem, three hypotheses must be satisfied:

1. Continuity on the Closed Interval [a, b]

The function must be continuous on the entire closed interval $[a, b]$. Continuity ensures no breaks, jumps, or asymptotes within the interval, which is crucial for the behavior of the function and the applicability of the theorem.

2. Differentiability on the Open Interval (a, b)

The function must be differentiable on the open interval (a, b). Differentiability implies the function has a well-defined tangent (derivative) at every point in (a, b), which is necessary for the existence of the point where the derivative equals zero.

3. Equal Function Values at the Endpoints: f(a) = f(b)

The function's values at the endpoints of the interval must be equal. This condition ensures the function starts and ends at the same height, creating a scenario where a maximum or minimum must occur somewhere within the interval, leading to a point where the slope is zero.

Steps to Verify the Hypotheses for a Given Function

Verifying that a function meets the hypotheses involves systematic checks:

Step 1: Verify Continuity on [a, b]

- Method: Analyze the function's form. Is it composed of continuous elementary functions (polynomials, exponentials, logarithms, trigonometric functions)?
- Potential issues: Discontinuities may arise from points where the function is undefined or has jumps (e.g., division by zero, asymptotes).
- Tools: Use the function's domain, limit analysis, and known properties of elementary functions.

Step 2: Verify Differentiability on (a, b)

- Method: Check the differentiability throughout the open interval. Is the function smooth? Are there any corners, cusps, or vertical tangents?
- Potential issues: Non-differentiability occurs at points where the function has sharp corners or vertical tangents.
- Tools: Derivative tests, geometric interpretation, or examining the function's derivative.

Step 3: Check that f(a) = f(b)

- Method: Evaluate the function at the endpoints. Are the values equal?
- Potential issues: If the function's endpoint values differ, Rolle's Theorem does not apply directly.

Illustrative Example: Applying Rolle's Theorem

Let's consider an example function to demonstrate the verification process.

Example: Verify whether Rolle's Theorem applies to $f(x) = x^3 - 3x + 2$ on the interval [1, 3].

Step 1: Check Continuity

- The function $f(x) = x^3 - 3x + 2$ is a polynomial.
- Polynomials are continuous everywhere, so $f(x)$ is continuous on [1, 3].

Step 2: Check Differentiability

- Since $f(x)$ is a polynomial, it is differentiable everywhere, including (1, 3).

Step 3: Check Endpoint Values

- $f(1) = 1^3 - 3(1) + 2 = 1 - 3 + 2 = 0$
- $f(3) = 3^3 - 3(3) + 2 = 27 - 9 + 2 = 20$
- Since $f(1) \neq f(3)$, the third hypothesis is not satisfied.
- Conclusion: Rolle's Theorem does not apply to $f(x)$ on [1, 3].

Now, consider the same function on the interval [1, 2].

- $f(1) = 0$
- $f(2) = 8 - 6 + 2 = 4$

Again, $f(1) \neq f(2)$, so no.

Alternatively, on the interval [0, 2]:

- $f(0)= 0 - 0 + 2= 2$
- $f(2)= 8 - 6 + 2= 4$

Still not equal, so Rolle's Theorem doesn't apply in these intervals.

Suppose we modify the function to $g(x) = \cos x$ on $[0, 2\pi]$:

- $g(0) = \cos 0=1$
- $g(2\pi) = \cos 2\pi=1$
- The function is continuous and differentiable everywhere, including $[0, 2\pi]$.

Therefore, the three hypotheses are satisfied:

- Continuous on $[0, 2\pi]$
- Differentiable on $(0, 2\pi)$
- $g(0) = g(2\pi)=1$

By Rolle's Theorem, there exists some $c \in (0, 2\pi)$ such that $g'(c) = 0$. Indeed, $g'(x) = -\sin x$, which equals 0 at $x=0, \pi, 2\pi$. Within $(0, 2\pi)$, $x=\pi$ satisfies the condition.

Additional Considerations and Common Pitfalls

While verifying the hypotheses seems straightforward, there are common pitfalls:

- **Ignoring the endpoint conditions:** Ensure that $f(a) = f(b)$. Many students mistakenly assume this without checking.
- **Overlooking the domain:** The domain must be carefully analyzed. For example, functions involving roots or logarithms may have restricted domains that affect continuity or differentiability.
- **Misidentifying points of non-differentiability:** Functions with sharp corners or vertical tangents at some points in (a, b) violate the differentiability hypothesis.

Extensions and Related Theorems

Once the hypotheses are verified, and Rolle's Theorem applies, it can be used to derive other important results:

Mean Value Theorem (MVT)

- The MVT generalizes Rolle's Theorem by dropping the condition $f(a) = f(b)$.

- It states that for functions continuous on [a, b] and differentiable on (a, b), there exists some c such that:

$$
f'(c) = \frac{f(b) - f(a)}{b - a}
$$

Applications of Rolle's Theorem

- Proving the existence of roots
- Establishing the behavior of polynomial functions
- Analyzing the symmetry of functions
- Justifying the existence of critical points

Conclusion

Verifying that a function satisfies the three hypotheses of Rolle's Theorem is a crucial step in many calculus problems. It involves analyzing the function's continuity, differentiability, and endpoint values. When these conditions are met, Rolle's Theorem assures the existence of at least one point within the interval where the derivative is zero, providing insights into the function's behavior and facilitating further analysis.

By systematically applying these checks and understanding the underlying principles, students and practitioners can confidently utilize Rolle's Theorem to solve a wide array of mathematical problems involving continuous and differentiable functions. Remember always to scrutinize the function's domain and properties to ensure the hypotheses are fully satisfied before drawing conclusions based on the theorem.

Frequently Asked Questions

What are the three hypotheses of Rolle's Theorem that a function must satisfy on a given interval?

The three hypotheses are: (1) The function is continuous on the closed interval [a, b], (2) the function is differentiable on the open interval (a, b), and (3) the function values at the endpoints are equal, i.e., f(a) = f(b).

How can I verify that a function is continuous on a specific interval when applying Rolle's Theorem?

To verify continuity, check if the function is continuous at every point within the interval [a, b], including the endpoints. Typically, this involves confirming that the function is defined at all points and that the limits from the left and right at each point match the function's value.

What steps should I take to confirm that a function is differentiable on an interval for Rolle's Theorem?

Determine if the function is differentiable at every point inside the interval (a, b). This usually involves calculating the derivative and checking for any points where the derivative does not exist or is undefined. For common functions, differentiability is often straightforward to verify.

Why is it important to check that the function values at the endpoints are equal when applying Rolle's Theorem?

Because Rolle's Theorem guarantees the existence of a point c in (a, b) where the derivative is zero only if the function values at the endpoints are equal. If $f(a) \neq f(b)$, the theorem does not apply, and the conclusion about the existence of such a point cannot be drawn.

If a function satisfies all three hypotheses of Rolle's Theorem on a given interval, what conclusion can be drawn?

The conclusion is that there exists at least one point c in the open interval (a, b) such that the derivative at that point, $f'(c)$, equals zero.

Can you provide an example of verifying the hypotheses of Rolle's Theorem for a specific function on an interval?

Yes. For example, consider $f(x) = x^2$ on $[-1, 1]$. It is continuous and differentiable everywhere, and $f(-1) = 1$, $f(1) = 1$, so the endpoint values are equal. Therefore, all hypotheses are satisfied, and Rolle's Theorem guarantees at least one c in $(-1, 1)$ with $f'(c) = 0$, which occurs at $c = 0$.

Additional Resources

Verify That The Function Satisfies The Three Hypotheses Of Rolle's Theorem On The Given Interval.

When approaching problems involving Rolle's Theorem, one of the most fundamental steps is to verify whether the function in question meets all three hypotheses of the theorem within the specified interval.

This process ensures that the theorem is applicable and that subsequent conclusions—such as the existence of at least one critical point where the derivative equals zero—are valid. This article offers a comprehensive guide to understanding, verifying, and interpreting the hypotheses of Rolle's Theorem, complete with detailed explanations, examples, and considerations.

Understanding Rolle's Theorem

What Is Rolle's Theorem?

Rolle's Theorem is a foundational result in differential calculus, providing a bridge between the behavior of a function and its derivatives. It states that if a function satisfies specific conditions over a closed interval, then there exists at least one point within that interval where the derivative of the function is zero.

Formal Statement:
Suppose f is a real-valued function satisfying the following three conditions on a closed interval $[a, b]$:

1. Continuity: f is continuous on $[a, b]$.
2. Differentiability: f is differentiable on (a, b).
3. Equal endpoints: $f(a) = f(b)$.

Then, there exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

Implications:
The theorem guarantees the existence of at least one horizontal tangent within the interval, which can be interpreted geometrically as the function having a peak, trough, or point of inflection where the slope is zero.

Step 1: Verifying Continuity

Why Is Continuity Essential?

The continuity of the function on $[a, b]$ ensures that the function has no jumps, breaks, or holes within the interval. This property is crucial because the theorem relies on the Intermediate Value Theorem, which requires continuity, to argue the existence of at least one point where the derivative is zero.

How to Verify Continuity

- Check the Function's Definition:
Ensure that f is defined for all x in $[a, b]$. If the function is given explicitly, examine its formula for potential points of discontinuity—such as divisions by zero, asymptotes, or points where the function is not defined.

- Assess Known Functions:
For standard functions like polynomials, sine, cosine, exponential, etc., continuity is guaranteed everywhere. For piecewise functions, verify that the function is continuous at the junction points by checking the limits from the left and right and ensuring they equal the function's value at those points.

- Use Theorems:
The Extreme Value Theorem states that continuous functions on closed intervals attain their maximum and minimum, which can be a quick indication of continuity if applicable.

Practical Example:
Suppose $f(x) = x^3 - 3x + 2$ on $[-1, 2]$. Since f is a polynomial, it is continuous everywhere, and on $[-1, 2]$, it is continuous.

Step 2: Verifying Differentiability

Why Is Differentiability Necessary?

Differentiability on (a, b) ensures that the function's derivative exists at every point inside the interval, which is essential for identifying the potential points where the derivative is zero.

How to Verify Differentiability

- Check the Function's Formula:
For explicitly given functions, compute the derivative and examine its domain. If the derivative exists at

all points in (a, b), the function is differentiable there.

- Identify Non-Differentiable Points:
Functions may fail to be differentiable at corners, cusps, vertical tangent lines, or points of discontinuity. Verify these points explicitly.

- Use Known Results:
Many standard functions are differentiable everywhere; for piecewise functions, ensure that the derivatives from the left and right agree at junctions.

Practical Example:
Continuing with $f(x) = x^3 - 3x + 2$, the derivative $f'(x) = 3x^2 - 3$ exists everywhere, confirming differentiability on $(-\infty, \infty)$.

Step 3: Checking the Endpoint Condition $f(a) = f(b)$

Why Is Equal Endpoint Value Important?

This condition ensures the function starts and ends at the same value, creating a "closed loop" where the function must, by the Intermediate Value Theorem, attain all intermediate values, including potentially a maximum or minimum where the slope is zero.

How to Verify $f(a) = f(b)$

- Calculate $f(a)$ and $f(b)$:
Direct substitution into the function's formula is often straightforward.

- Compare the Values:
Confirm whether $f(a)$ and $f(b)$ are equal within a reasonable tolerance if dealing with approximations.

- Consider the Significance:
If $f(a) \neq f(b)$, Rolle's Theorem does not directly apply. However, the Mean Value Theorem may be relevant, which relaxes this condition.

Practical Example:

For $f(x) = \cos x$ over $[0, 2\pi]$, $f(0) = 1$ and $f(2\pi) = 1$. Since these are equal, the endpoint condition is satisfied.

Additional Considerations and Common Pitfalls

Ensuring All Conditions Are Met

While verifying the hypotheses, keep in mind:

- The interval must be closed $[a, b]$.
- The function must be continuous on the entire interval, not just open or half-open.
- The function must be differentiable on the open interval (a, b).
- The endpoint values must match exactly.

Potential Errors to Watch For:
- Overlooking discontinuities or non-differentiable points.
- Miscalculating endpoint values.
- Applying Rolle's Theorem to functions that fail one of the hypotheses.

Examples of Functions That Fail Hypotheses

- $f(x) = |x|$ on $[-1, 1]$: Not differentiable at $x=0$.
- $f(x) = 1/x$ on $[1, 2]$: Not continuous at $x=1$ (if considering the limit from the left).
- $f(x) = x^2 + 1$ on $[0, 1]$: $f(0) = 1$, $f(1) = 2$, so the endpoint condition fails.

Interpreting the Verification Results

Once all three hypotheses are confirmed, Rolle's Theorem applies, and you can confidently state that there exists some $c \in (a, b)$ such that $f'(c) = 0$. This point often corresponds to a local maximum, minimum, or an inflection point, depending on the function's shape.

Conversely, if any of the hypotheses are not satisfied, then the theorem does not hold, and the conclusion about the existence of such a point cannot be guaranteed.

Practical Applications and Extensions

- Finding Critical Points:
Verifying hypotheses is the first step in locating points where the derivative is zero, which are essential for optimization problems.

- Connecting to the Mean Value Theorem:
When $f(a) \neq f(b)$, the Mean Value Theorem extends the idea, guaranteeing a point where the derivative equals the average rate of change.

- Generalizations:
Rolle's Theorem is a special case of the Mean Value Theorem, and understanding its hypotheses aids in grasping broader differential calculus concepts.

Summary of Steps to Verify the Hypotheses

- Continuity on $[a, b]$:
Confirm the function has no jumps or breaks.

- Differentiability on (a, b):
Ensure the derivative exists throughout the open interval.

- Equal endpoint values:
Calculate $f(a)$ and $f(b)$ to verify they are equal.

Conclusion

Verifying that a function satisfies the three hypotheses of Rolle's Theorem is a crucial preliminary step in many calculus problems. It ensures the validity of subsequent deductions about the existence of points where the function's derivative is zero. By carefully analyzing continuity, differentiability, and endpoint values, one can confidently apply Rolle's Theorem to gain insights into the behavior of functions and their critical points. Mastery of this verification process not only simplifies problem-solving but also deepens understanding of fundamental calculus principles.

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