What Is The Distance Along The Unit Circle Between Any Two Successive 8th Roots Of 1?a. /8b. /6c. /4d.

What Is The Distance Along The Unit Circle Between Any Two Successive 8th Roots Of 1?a. /8b. /6c. /4d.
Understanding the distance between successive roots of unity on the unit circle is a fundamental concept in complex analysis and trigonometry. Specifically, for the 8th roots of unity, which are points evenly spaced around the unit circle in the complex plane, determining the exact distance between any two adjacent roots provides insight into the geometric structure of roots of unity. This article explores how to calculate this distance, delves into the mathematical reasoning behind it, and compares it with roots of other orders such as 6th and 4th roots of unity.

Introduction to Roots of Unity

Roots of unity are solutions to the equation $z^n = 1$, where n is a positive integer. These solutions are complex numbers evenly spaced on the unit circle in the complex plane. The concept of roots of unity is essential in various fields including algebra, number theory, signal processing, and Fourier analysis.

What Are the 8th Roots of Unity?

The 8th roots of unity are solutions to the equation:
$$ z^8 = 1 $$
They can be expressed as:
$$ z_k = e^{2\pi i k / 8} \quad \text{for} \quad k=0,1,2,\dots,7 $$
which simplifies to:
$$ z_k = e^{i \pi k / 4} $$
Each root corresponds to a point on the unit circle at an angle of $\frac{\pi k}{4}$ radians from the positive real axis.

The Geometric Representation of 8th Roots of Unity

On the complex plane, the 8th roots of unity are equally spaced points around the circle of radius 1. They form a regular octagon inscribed within the circle.

Key Properties of 8th Roots of Unity

- Equal Spacing: The roots are equally spaced at angles of $\frac{\pi}{4}$ radians (or 45 degrees).
- Symmetry: They exhibit rotational symmetry of order 8.
- Complex Numbers: Each root can be represented as $\cos \frac{\pi k}{4} + i \sin \frac{\pi k}{4}$.

Calculating the Distance Between Successive 8th Roots of Unity

The core question is: What is the distance along the unit circle between any two successive 8th roots of unity? This is a geometric problem involving the chord length between two points on a circle separated by a fixed angle.

The Formula for Chord Length

Given two points on a circle separated by an angle θ, the length of the chord connecting them is:

$$\text{Chord length} = 2r \sin \left(\frac{\theta}{2} \right)$$

where r is the radius of the circle. Since we're dealing with the unit circle, $r=1$, so the formula simplifies to:

$$\text{Chord length} = 2 \sin \left(\frac{\theta}{2} \right)$$

Applying the Formula to 8th Roots of Unity

The angular separation between two successive 8th roots is:

$$\theta = \frac{2\pi}{8} = \frac{\pi}{4}$$

Therefore, the distance d between any two neighboring roots is:

$$d = 2 \sin \left(\frac{\pi/4}{2} \right) = 2 \sin \left(\frac{\pi}{8} \right)$$

Calculating $2 \sin \left(\frac{\pi}{8} \right)$

To find the exact numerical value, recall that:

$$\sin \left(\frac{\pi}{8} \right) = \sin 22.5^\circ$$

which can be derived using half-angle formulas.

Half-Angle Formula for Sine

$$ \sin \frac{\theta}{2} = \pm \sqrt{ \frac{1 - \cos \theta}{2} } $$
Applying this to $ \theta = \frac{\pi}{4} $:
$$ \sin \frac{\pi}{8} = \sqrt{ \frac{1 - \cos \frac{\pi}{4}}{2} } $$

Since $ \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} $, we have:
$$ \sin \frac{\pi}{8} = \sqrt{ \frac{1 - \frac{\sqrt{2}}{2}}{2} } $$

Simplify numerator:
$$ 1 - \frac{\sqrt{2}}{2} = \frac{2 - \sqrt{2}}{2} $$

Therefore:
$$ \sin \frac{\pi}{8} = \sqrt{ \frac{ \frac{2 - \sqrt{2}}{2} }{2} } = \sqrt{ \frac{2 - \sqrt{2}}{4} } $$

Expressed as:
$$ \sin \frac{\pi}{8} = \frac{\sqrt{2 - \sqrt{2}}}{2} $$

Finally, the distance between successive roots is:
$$ d = 2 \times \frac{\sqrt{2 - \sqrt{2}}}{2} = \sqrt{2 - \sqrt{2}} $$

Summary of the Distance Between Successive 8th Roots of Unity

The exact distance along the unit circle between any two successive 8th roots of unity is:
$$ \boxed{ \sqrt{2 - \sqrt{2}} } $$

This value is approximately 0.765, indicating that each adjacent root is separated by this chord length along the circle.

Comparison with Roots of Other Orders

Understanding the distance between roots of unity of different orders offers deeper insights into their geometric arrangements.

6th Roots of Unity

- Angular separation: $ \frac{2\pi}{6} = \frac{\pi}{3} $ radians (60 degrees).
- Chord length: $ 2 \sin \left(\frac{\pi/3}{2} \right) = 2 \sin \left(\frac{\pi}{6} \right) = 2 \times \frac{1}{2} = 1 $.

- Interpretation: Successive 6th roots are separated by a chord of length 1 on the unit circle.

4th Roots of Unity

- Angular separation: $\frac{2\pi}{4} = \frac{\pi}{2}$ radians (90 degrees).
- Chord length: $2 \sin \left(\frac{\pi/2}{2} \right) = 2 \sin \left(\frac{\pi}{4} \right) = 2 \times \frac{\sqrt{2}}{2} = \sqrt{2}$.
- Interpretation: Adjacent roots are separated by a chord length of $\sqrt{2}$.

Applications and Significance of These Calculations

Calculating distances between roots of unity is not just a theoretical exercise; it has practical applications in various fields.

Applications in Signal Processing and Fourier Analysis

- Roots of unity underpin the Discrete Fourier Transform (DFT), which decomposes signals into frequency components.
- Understanding their geometric properties allows for efficient algorithm design.

Applications in Algebra and Number Theory

- Roots of unity are used in cyclotomic polynomials and field extensions.
- Distances between roots relate to polynomial factorization and Galois groups.

Applications in Computer Graphics and Geometry

- Regular polygons inscribed in circles are constructed using roots of unity.
- Distances between roots determine side lengths and symmetry properties.

Conclusion

The distance along the unit circle between any two successive 8th roots of unity is exactly $\sqrt{2 - \sqrt{2}}$. This value stems directly from the chord length formula, considering the angle of $\frac{\pi}{4}$ radians between successive roots. Understanding this geometric relationship enhances comprehension of roots of unity and their applications in mathematics, engineering, and computer science.

By extending this analysis to roots of other orders, such as 6th and 4th roots, we see how the spacing varies systematically, governed by the sine function and geometric principles. The study of roots of unity reveals the elegant harmony between algebra and geometry, illustrating how complex numbers encode symmetries and structures fundamental to many scientific disciplines.

Frequently Asked Questions

What is the main concept behind finding the distance between successive 8th roots of unity on the unit circle?

The main concept involves understanding the angular separation between consecutive roots, which are evenly spaced around the circle, and then calculating the chord length between those points.

How are the 8th roots of unity represented on the complex plane?

They are represented as points on the unit circle at equal angles of 45° ($\pi/4$ radians) apart, corresponding to their roots of unity.

What is the formula for the distance between two successive 8th roots of unity?

The distance is given by $2 \times \sin(\pi/8)$, since the chord length between two points separated by an angle of $\pi/4$ radians on a unit circle is $2 \times \sin(\pi/8)$.

What is the approximate numerical value of the distance between two successive 8th roots of unity?

Approximately 0.765, since $2 \times \sin(\pi/8) \approx 2 \times 0.3827 \approx 0.765$.

Which option (a, b, c, d) corresponds to the distance between successive 8th roots of unity?

Option a. /8 is correct because the distance is $2 \times \sin(\pi/8)$, which simplifies to $(\sqrt{2} - 1)$.

Why is the distance between successive roots of unity the same for all pairs on the circle?

Because the roots are evenly spaced around the circle, the angular separation is constant, resulting in equal chord lengths between each pair.

How does the angle between successive 8th roots relate to the unit circle?

The angle is 45° ($\pi/4$ radians), which determines the chord length between successive roots via the sine function.

Can the distance between successive 8th roots of unity be derived using trigonometry?

Yes, by using the sine of half the angle between roots, the distance is $2 \times \sin(\pi/8)$.

What is the significance of understanding the distance between roots on the unit circle?

It helps in understanding the geometric distribution of roots of unity, which is fundamental in fields like Fourier analysis, signal processing, and complex number theory.

Additional Resources

Distance Along the Unit Circle Between Successive 8th Roots of Unity

Understanding the geometric and algebraic properties of roots of unity is fundamental in complex analysis, signal processing, and numerous fields of mathematics and engineering. A particularly intriguing aspect is determining the distance along the unit circle between successive roots, especially for roots of different orders such as eighth, sixth, or fourth roots of unity. This article delves deep into the concept, exploring the mathematical foundations, derivations, and implications of the distances between successive roots on the unit circle for various roots of unity.

Introduction to Roots of Unity and the Unit Circle

What Are Roots of Unity?

In complex analysis, roots of unity are solutions to the equation:

$$ z^n = 1 $$

where n is a positive integer. These solutions are complex numbers z that, when raised to the n-th power, equal 1. The set of all n-th roots of unity is fundamental because they are evenly spaced points on the complex plane, lying on the unit circle (the circle with radius 1 centered at the origin).

For example, the n-th roots of unity are given by:

$$ z_k = e^{2\pi i k / n} \quad \text{for} \quad k = 0, 1, 2, ..., n-1 $$

where $e^{i\theta} = \cos \theta + i \sin \theta$.

The Unit Circle and Roots of Unity

The unit circle in the complex plane is the set:

$$ \{ z \in \mathbb{C} : |z| = 1 \} $$

The roots of unity are points on this circle at equally spaced angles:

$$ \theta_k = \frac{2\pi k}{n} $$

for each k. Visually, these roots form a regular polygon inscribed in the circle, with each vertex representing one root.

Understanding the Distance Between Successive Roots

What Does "Distance Along the Unit Circle" Mean?

The phrase refers to the arc length on the circle between two successive roots. Since roots are evenly distributed around the circle, the arc between two adjacent roots corresponds to a fixed angle, and the length of this arc depends on the circle's radius and the angle between the points.

Given that all roots of unity lie on the unit circle, the radius $r = 1$. The distance along the circle (arc length) between two successive roots is thus:

$$\text{arc length} = r \times \text{central angle}$$

Because $r = 1$, the arc length simplifies to:

$$\text{arc length} = \text{central angle}$$

(where the central angle is in radians).

Calculating the Central Angle Between Successive Roots

For roots of order n, the roots are separated by an angle:

$$\Delta \theta = \frac{2\pi}{n}$$

This is the angle between two successive roots z_k and z_{k+1}.

Therefore, the arc length (distance along the circle) between successive roots is:

$$\boxed{\text{Distance} = \frac{2\pi}{n}}$$

since the radius of the unit circle is 1.

Specific Cases: Distances for Roots of Different Orders

Having established that the arc length between successive roots of unity is $2\pi / n$, we can now analyze the specific cases for roots of orders 8, 6, and 4.

a. Eighth Roots of Unity

- Order: $n = 8$
- Number of roots: 8
- Angular separation: $2\pi / 8 = \pi / 4$

Arc length between successive roots:

$$\frac{2\pi}{8} = \frac{\pi}{4}$$

Interpretation:

This means that each successive 8th root of unity is separated by an arc length of $\pi/4$ radians (or 45 degrees). If you imagine the unit circle, moving from one root to the next along the circle covers an arc of $\pi/4$. In linear distance (the actual length of the arc):

$$\text{Arc length} = r \times \Delta \theta = 1 \times \frac{\pi}{4} = \frac{\pi}{4}$$

Additional Insight:

- The chord length between successive roots (the straight-line distance connecting two adjacent roots) can be calculated using the Law of Cosines or the formula:

$$\text{Chord length} = 2r \sin \left(\frac{\Delta \theta}{2} \right) = 2 \times 1 \times \sin \left(\frac{\pi}{8} \right)$$

- Numerically:

$$2 \sin \left(\frac{\pi}{8} \right) \approx 2 \times 0.3827 \approx 0.7654$$

b. Sixth Roots of Unity

- Order: $n = 6$
- Number of roots: 6
- Angular separation: $2\pi / 6 = \pi / 3$

Arc length between successive roots:

$$\frac{2\pi}{6} = \frac{\pi}{3}$$

Interpretation:

Each successive 6th root is separated by an arc length of $\pi/3$ radians (or 60 degrees). The straight-line chord length between roots:

$$ 2 \sin \left(\frac{\pi}{6} \right) = 2 \times 0.5 = 1 $$

which is a significant distance, indicating the roots are relatively spaced out.

c. Fourth Roots of Unity

- Order: $n = 4$
- Number of roots: 4
- Angular separation: $2\pi / 4 = \pi / 2$

Arc length between successive roots:

$$ \frac{2\pi}{4} = \frac{\pi}{2} $$

Interpretation:

Successive 4th roots are separated by an arc length of $\pi/2$ radians (or 90 degrees). The chord length between these roots:

$$ 2 \sin \left(\frac{\pi}{4} \right) = 2 \times \frac{\sqrt{2}}{2} = \sqrt{2} \approx 1.4142 $$

Mathematical Derivation of the Distance Between Successive Roots

From the Roots' Positions on the Complex Plane

Given roots:

$$ z_k = e^{2\pi i k / n} $$

for $k = 0, 1, ..., n-1$, the successive roots are:

$$ z_k = e^{2\pi i k / n} $$

and

$$ z_{k+1} = e^{2\pi i (k+1) / n} $$

The distance between these two roots in the complex plane is given by their difference:

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\[ |z_{k+1} - z_k| \]

Calculating:

\[
\begin{aligned}
|z_{k+1} - z_k| &= |e^{2\pi i (k+1)/n} - e^{2\pi i k / n}| \\
&= \left| e^{2\pi i k / n} (e^{2\pi i / n} - 1) \right| \\
&= |e^{2\pi i k / n}| \times |e^{2\pi i / n} - 1| \\
&= 1 \times |e^{2\pi i / n} - 1| \quad \text{(since } |e^{i\theta}|=1) \\
&= |e^{i \theta} - 1| \quad \text{where } \theta = \frac{2\pi}{n}
\end{aligned}
\]

Expressed in terms of sine and cosine:

\[
|e^{i \theta} - 1| = \sqrt{(\cos \theta - 1)^2 + (\sin \theta)^2}
\]

Simplify:

\[
\begin{aligned}
&= \sqrt{ (\cos \theta - 1)^2 + \sin^2 \theta } \\
&= \sqrt{ \cos^2 \theta - 2 \cos \theta + 1 + \sin^2 \theta } \\
&= \sqrt{ (\cos^2 \theta + \sin^2 \theta) - 2 \cos \theta + 1 } \\
&= \sqrt{ 1 - 2 \cos \theta + 1 } \\
&= \sqrt{ 2 - 2 \cos \theta } \\
&= \sqrt{
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