A) Determine Which Amounts Of Postage Can Be Formed Using Just 4-cent And 11-cent Stamps. B) Prove Your

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In the realm of combinatorial mathematics and number theory, problems involving the formation of specific amounts using limited denominations of coins or stamps are both classic and intellectually stimulating. The problem at hand is to identify which postage amounts can be formed exclusively using 4-cent and 11-cent stamps. Furthermore, we are tasked with providing a rigorous proof of our findings, demonstrating a thorough understanding of the underlying principles. This article explores the problem through systematic analysis, mathematical reasoning, and proof techniques such as the Chicken McNuggets theorem, also known as the Frobenius coin problem, to arrive at definitive conclusions.

Understanding the Problem

Restating the Question

The goal is to determine all positive integer amounts of postage that can be exactly assembled using only 4-cent and 11-cent stamps. Formally, for a given amount N, we want to know whether there exist non-negative integers x and y such that:

- $N = 4x + 11y$
- $x \geq 0$, $y \geq 0$

Key Definitions and Concepts

Before delving into the solution, it is essential to clarify some concepts:

- **Linear combinations:** Expressions of the form $N = a x + b y$, where a and b are fixed positive integers, and x, y are non-negative integers.

- **Coin (or stamp) denomination:** The fixed values of stamps available—in this case, 4 and 11 cents.
- **Representability:** The ability to express a given amount as a linear combination of the denominations with non-negative integer coefficients.

Mathematical Background and Theoretical Tools

The Frobenius Coin Problem

The Frobenius coin problem, also known as the Chicken McNuggets theorem in the special case of two denominations, precisely addresses the question of which amounts can be formed using two positive integers that are coprime.

For two positive integers a and b that are coprime (i.e., their greatest common divisor (gcd) is 1), the Frobenius number g(a, b) is the largest integer that cannot be expressed as a non-negative linear combination of a and b. All integers greater than g(a, b) can be expressed as such a combination.

Applying the Frobenius Number Theorem

In our case, the two denominations are 4 and 11. To apply the Frobenius number theorem directly, we need to verify whether 4 and 11 are coprime:

- gcd(4, 11) = 1

Since 4 and 11 are coprime, the Frobenius number g(4, 11) is given by:

g(4, 11) = (a b) - a - b = (4 11) - 4 - 11 = 44 - 4 - 11 = 29

This means:

- The largest amount that cannot be formed using any non-negative combinations of 4 and 11 is 29 cents.
- All amounts greater than 29 can be formed using some combination of 4-cent and 11-cent stamps.

Determining Which Amounts Are Possible

Implications of the Frobenius Number

From the above, we know that:

- Any amount greater than 29 cents can be formed using 4-cent and 11-cent stamps.
- Amounts less than or equal to 29 need to be checked individually to determine if they are representable.

Listing the Impossible Amounts

Because the Frobenius number $g(4, 11) = 29$, the only amounts that cannot be formed are those less than or equal to 29 and not expressible as a linear combination of 4 and 11. To identify these, we analyze amounts from 1 to 29 to find which are impossible.

Methodology for Verification

We systematically check each amount N from 1 to 29:

1. Try to express N as $4x + 11y$ with $x, y \geq 0$.
2. If such x and y exist, N is representable; otherwise, it is not.

Explicit Analysis for N = 1 to 29

Impossible Amounts

Let's determine which amounts cannot be expressed as $4x + 11y$.

Amounts less than 4:

- 1, 2, 3: Cannot be formed (since the smallest stamp is 4 cents).

Amounts from 4 to 29:

We check each amount:

- **4:** 4 = 4×1 + 11×0 → possible
- **5:** Cannot be formed (since 5 - 4 = 1, which cannot be formed)
- **6:** Cannot be formed (6 - 4 = 2, cannot be formed)
- **7:** Cannot be formed (7 - 4 = 3, cannot be formed)
- **8:** 8 = 4×2 + 11×0 → possible
- **9:** 9 - 4 = 5 (impossible), 9 - 11 = negative, so impossible
- **10:** 10 = 4×2 + 11×0 → possible
- **11:** 11 = 11×1 + 4×0 → possible
- **12:** 12 = 4×3 + 11×0 → possible
- **13:** 13 - 11 = 2 (impossible), 13 - 4 = 9 (impossible), so impossible
- **14:** 14 = 11 + 4×1 → possible
- **15:** 15 - 11 = 4, so 15 = 11 + 4 → possible
- **16:** 16 = 4×4 → possible
- **17:** 17 - 11 = 6 (impossible), so check other combinations; 17 - 4×3 = 17 - 12 = 5 (impossible), so impossible
- **18:** 18 = 4×4 + 11×0 → possible
- **19:** 19 - 11 = 8, which is possible (8=4×2), so 19 = 11 + 4×2 → possible
- **20:** 20 = 4×5 → possible
- **21:** 21 = 11 + 4×2 → possible
- **22:** 22 = 11×2 → possible
- **23:** 23 - 11 = 12 (possible), so 23 = 11 + 12 → 12 is possible, so 23 = 11 + 4×3 → possible
- **24:** 24 = 4×6 → possible
- **25:** 25 - 11 = 14 (possible), so 25 = 11 + 14 → 14 is possible, so possible

- **26:** 26 = 11 + 15 → 15 is possible, so 26 = 11 + 4×3 → possible
- **27:** 27 - 11 = 16 (possible), so 27 = 11 + 16 → possible
- **28:** 28 = 4×7 → possible
- **29:** 29 - 11 = 18 (possible), so 29 = 11 + 18 → possible

Summary

Frequently Asked Questions

How can I determine which postage amounts can be formed using only 4-cent and 11-cent stamps?

You can model the problem using the Frobenius coin problem, which involves finding all amounts that can be expressed as a linear combination of 4 and 11 with non-negative integers. Since 4 and 11 are coprime, all amounts greater than or equal to (411 - 4 - 11) = 29 cents can be formed, and smaller amounts can be checked individually.

What is the Frobenius number for stamps of 4 and 11 cents, and what does it signify?

The Frobenius number for 4 and 11 is 29, meaning that 29 cents is the largest amount that cannot be formed using any combination of 4-cent and 11-cent stamps. Any amount greater than 29 can be formed with these stamps.

Can all amounts less than 29 cents be formed using just 4-cent and 11-cent stamps?

No, not all less than 29 cents can be formed. For example, amounts like 1, 2, 3, 5, 6, 7, 9, 10, 13, etc., cannot be expressed as a sum of 4s and 11s. These need to be checked individually.

How do I prove which specific amounts cannot be formed with 4 and 11-cent stamps?

You can systematically test each amount by attempting to express it as 4a + 11b, where a and b are non-negative integers. If no such combination exists, that amount cannot be formed. Alternatively, use modular arithmetic to identify impossible amounts.

What pattern emerges for amounts that can be formed with 4-cent and 11-cent stamps?

Once past the Frobenius number (29 cents), all larger amounts can be formed. For smaller amounts, only specific combinations work, but a pattern shows that amounts congruent to certain residues mod 4 or 11 are impossible, helping to identify which amounts are unattainable.

Can the problem be generalized for other coin denominations, and how?

Yes, the Frobenius coin problem can be generalized to any two coprime positive integers. The Frobenius number is calculated as (a b) - a - b, and amounts greater than this can be formed using combinations of these denominations.

What is the significance of proving which amounts can be formed using just 4 and 11-cent stamps?

Proving this helps understand the limitations of combinatorial sums, aids in solving real-world postal denomination problems, and demonstrates important concepts in number theory related to linear Diophantine equations and the Frobenius coin problem.

How can I use these concepts to determine postage options in practical scenarios?

By understanding which amounts are achievable with given stamp denominations, you can optimize your postage purchases, ensure accurate payment for specific amounts, and develop algorithms for automated postage calculation based on available stamp denominations.

Additional Resources

A) Determine Which Amounts Of Postage Can Be Formed

Using Just 4-cent And 11-cent Stamps. B) Prove Your

In the world of postal services and mathematical puzzles alike, the question of which postage amounts can be formed using limited denominations is both intriguing and challenging. Postal workers, philatelists, and mathematicians have long grappled with the problem: Given a set of stamp denominations, which total amounts can be assembled using some combination of those stamps? This question becomes particularly engaging when the denominations are as small and seemingly simple as 4 cents and 11 cents.

In this article, we will explore the problem in depth, starting with a clear determination of which amounts of postage can be formed using only 4-cent and 11-cent stamps. We will then rigorously prove our findings, illustrating how number theory and the concept of linear Diophantine equations provide the tools needed to solve this problem. Whether you’re a mathematics enthusiast, a postal worker, or a curious reader, this journey will shed light on a classic problem that blends practical constraints with elegant theory.

Understanding the Problem: The Core Question

At its core, the problem asks:

Given only 4-cent and 11-cent stamps, which total postage amounts can be created by combining some number of these stamps?

This question can be reframed as a mathematical problem:

> Find all non-negative integer solutions (x, y) to the equation:
>
> $4x + 11y = N$
>
> where N is the total postage amount, and $x, y \geq 0$.

The core goal is to determine the set of all possible N for which this equation has solutions.

Mathematical Foundations: Linear Combinations and the Frobenius Coin Problem

To analyze which amounts can be formed, it’s essential to understand the underlying mathematical principles. The problem aligns with a classic area of number theory called the Coin Problem or Frobenius Problem, which deals with expressing integers as linear combinations of given positive integers.

Key Concepts:

- Linear combinations: Any amount N can be expressed as a combination of the denominations if integers x, y ≥ 0 satisfy the equation.
- Greatest Common Divisor (gcd): The set of all amounts that can be formed using denominations a and b depends critically on their gcd.

The Frobenius Number is the largest amount that cannot be formed using two positive integer denominations that are coprime (share no common divisors other than 1). For two denominations a and b, the Frobenius number G is given by:

$$G = a b - a - b$$

All amounts larger than G can be formed using a and b.

Applying Theory to Our Denominations: 4 cents and 11 cents

Given our specific denominations:

- a = 4
- b = 11

First, determine their gcd:

$$\gcd(4, 11) = 1$$

Since the gcd is 1, we know that sufficiently large amounts can be formed using some combination of 4-cent and 11-cent stamps.

Calculating the Frobenius Number:

$G = (4)(11) - 4 - 11 = 44 - 4 - 11 = 29$

This means:

- All amounts greater than 29 cents can be formed using some combination of 4-cent and 11-cent stamps.
- Certain amounts less than or equal to 29 cents may or may not be formable, and these need to be checked individually.

Determining the Reachable Amounts: Which Postage Values Can Be Formed?

Based on the Frobenius number, the following conclusions emerge:

1. All amounts greater than 29 cents are achievable.
2. Amounts less than or equal to 29 cents require individual verification.

Let's proceed to verify which amounts up to 29 cents can be formed:

- 4 cents: $4 \times 1 = 4 \rightarrow$ achievable
- 8 cents: $4 \times 2 = 8 \rightarrow$ achievable
- 11 cents: $11 \times 1 = 11 \rightarrow$ achievable
- 12 cents: $4 \times 3 = 12 \rightarrow$ achievable
- 15 cents: $4 \times 1 + 11 \times 1 = 15 \rightarrow$ achievable
- 16 cents: $4 \times 4 = 16 \rightarrow$ achievable
- 19 cents: $4 \times 2 + 11 \times 1 = 19 \rightarrow$ achievable
- 20 cents: $4 \times 5 = 20 \rightarrow$ achievable
- 22 cents: $11 \times 2 = 22 \rightarrow$ achievable
- 23 cents: $4 \times 1 + 11 \times 2 = 4 + 22 = 26 \rightarrow$ too low; check other combinations:

$4 \times 0 + 11 \times 2 = 22$ (no),

$4 \times 1 + 11 \times 1 = 15$,

$4 \times 2 + 11 \times 1 = 19$,

$4 \times 3 + 11 \times 0 = 12$,

$4 \times 4 + 11 \times 0 = 16$,

$4 \times 0 + 11 \times 2 = 22$,

$4 \times 1 + 11 \times 1 = 15$,

$4 \times 2 + 11 \times 1 = 19$,

$4 \times 3 + 11 \times 1 = 23 \rightarrow$ Yes, 23 cents is achievable.

- 25 cents: $4 \times 1 + 11 \times 2 = 4 + 22 = 26$ (No),

$4 \times 2 + 11 \times 1 = 8 + 11 = 19$,

$4 \times 3 + 11 \times 1 = 12 + 11 = 23$,

$4 \times 4 + 11 \times 1 = 16 + 11 = 27$,

$4 \times 0 + 11 \times 2 = 22$,

$4 \times 5 + 11 \times 0 = 20$,

$4 \times 2 + 11 \times 1 = 19$,

No combination equals 25. So 25 cents cannot be formed.

- 26 cents: as shown, $4 \times 2 + 11 \times 2 = 8 + 22 = 30$ (no), check other combinations:

$4 \times 1 + 11 \times 2 = 4 + 22 = 26 \rightarrow$ Yes, achievable.

Similarly, we continue this process for each amount up to 29 cents.

Summary of achievable amounts (up to 29 cents):

- Achievable: 4, 8, 11, 12, 15, 16, 19, 20, 22, 23, 26, 27, 28, 29 (after thorough checking)
- Not achievable: 5, 6, 7, 9, 10, 13, 14, 17, 18, 21, 24, 25

In practice, what this demonstrates is that:

- The set of amounts that cannot be formed are finite and include specific small amounts.
- Beyond 29 cents, every amount can be formed with some combination of 4 and 11.

Formal Proof: Using Number Theory to Confirm the Results

Theorem:
Since gcd(4, 11) = 1, every sufficiently large integer $N \geq G+1 = 30$ can be expressed as a non-negative integer combination of 4 and 11.

Proof Sketch:

1. Existence of solutions for large N:
Because 4 and 11 are coprime, the set of possible sums includes all integers greater than the Frobenius number (29). This is a standard result in number theory.

2. Constructive proof for small N:
For each $N \leq 29$, explicitly check whether solutions exist or not, as demonstrated above.

3. Conclusion:
All integers greater than 29 are achievable, and only a finite set of small amounts are not.

This confirms that the postage amounts that can be formed using only 4-cent and 11-cent stamps are all integers ≥ 30, plus the specific smaller amounts that can be constructed explicitly.

Implications and Practical Applications

This mathematical insight is more than a theoretical curiosity—it has practical implications for postal services and stamp collectors:

- Designing stamp denominations:
Postal authorities can determine which postage values are impossible to form with given denominations, helping avoid inefficiencies or shortages.

- Philatelic challenges:
Collectors and hobbyists can test their problem-solving skills against these classic sums, knowing precisely which amounts are achievable.

- Educational value:
The problem illustrates fundamental concepts of number theory, such as gcd, linear Diophantine equations, and the Frobenius number, providing a practical context for abstract mathematical ideas.

Conclusion: The Takeaway

The question of which postage amounts can be formed using just 4-cent and 11-cent stamps reveals deep connections between simple arithmetic and advanced number theory. The key points are:

- Since 4 and 11 are coprime, all amounts greater than 29 cents can be formed by some combination of these stamps.
- There exists a finite set of smaller amounts that cannot be assembled using these denominations, identified through explicit checking.
- The Frobenius number (29) acts as a

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