A Fair Die Is Rolled 14 Times. Let Be The Number Of Faces That Appear Exactly Three Times. Which Of The

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When analyzing probability scenarios involving dice, understanding the likelihood of specific outcomes is essential for both enthusiasts and professionals in the fields of statistics, mathematics, and gaming. One interesting problem involves rolling a fair six-sided die 14 times and determining the probability distribution of faces that appear exactly three times. This problem combines combinatorial concepts, probability theory, and statistical reasoning to explore the possible configurations and their likelihoods.

In this comprehensive guide, we'll delve deeply into this problem, breaking it down into manageable sections. We'll review the fundamental principles of probability theory relevant to dice rolls, explore the combinatorial calculations involved, and analyze the expected outcomes. Whether you're a student preparing for examinations, a game designer considering probabilities, or a hobbyist interested in mathematical puzzles, this article provides a detailed, structured exploration of the problem.

Understanding the Problem

The Scenario

- You roll a fair six-sided die 14 times.
- Each roll results in one face from 1 to 6.
- The goal is to analyze the distribution of faces that appear exactly three times.

The Key Question

Given this setup, which face(s) are likely to appear exactly three times? Additionally, how many faces can appear exactly three times, and what is the probability distribution of such outcomes?

Fundamental Concepts in Probability and Combinatorics

Before addressing the specific problem, it's essential to review some core concepts that underpin the analysis:

1. Probability of a Single Outcome

- For a fair six-sided die, the probability that any particular face appears on a single roll is \(\frac{1}{6} \).

- The probability of a sequence of outcomes depends on the individual probabilities, assuming independence.

2. Multinomial Distribution

- When multiple independent trials are conducted with multiple possible outcomes, the multinomial distribution describes the probability of counts for each outcome.
- For dice rolls, the number of times each face appears over 14 rolls follows a multinomial distribution with parameters $n = 14$ and probabilities $p_i = \frac{1}{6}$ for each face.

3. Combinations and Permutations

- To count the number of sequences where certain faces appear a specific number of times, combinatorial calculations are necessary.
- The total number of possible sequences is 6^{14}, since each of the 14 rolls can be any of 6 faces.

Breaking Down the Problem: Faces Appearing Exactly Three Times

1. What Does "Faces That Appear Exactly Three Times" Mean?

- For the 14 rolls, some subset of faces will occur exactly three times.
- The remaining rolls are distributed among other faces, which may appear more or fewer times, but we're focusing on faces with a count of exactly three appearances.

2. Possible Numbers of Faces That Appear Exactly Three Times

- Since each face that appears exactly three times accounts for 3 rolls, the maximum number of such faces is limited by $14 \div 3$, because:

$$
\text{Maximum number of faces appearing exactly three times} = \left\lfloor \frac{14}{3} \right\rfloor = 4
$$

- However, because 4 faces appearing exactly three times sum to 12 rolls, the remaining 2 rolls are distributed among other faces.

- Therefore, the possible counts of faces appearing exactly three times are:

$$
k = 0, 1, 2, 3, 4
$$

with the understanding that each configuration must fit into the total of 14 rolls.

Calculating the Number of Faces That Appear Exactly Three Times

1. General Approach

To determine the number of faces appearing exactly three times, we need to:

- Choose which faces will appear exactly three times.
- Assign counts to the remaining faces, ensuring the total adds up to 14.
- Count the number of sequences (outcomes) consistent with these counts.

2. Step-by-Step Calculation

Let's formalize this process:

Step 1: Choose the Faces That Appear Exactly Three Times

- For k faces to appear exactly three times, select k faces out of 6:

$$
\text{Number of ways} = \binom{6}{k}
$$

Step 2: Assign Counts to the Selected Faces

- Each of these k faces appears exactly 3 times, contributing $3k$ rolls.

- Remaining $14 - 3k$ rolls are distributed among the remaining $6 - k$ faces.

- These remaining counts can vary, but for each configuration, the counts must be positive integers (or zero) summing to $14 - 3k$.

Step 3: Distribute Remaining Rolls

- For the remaining $6 - k$ faces, the counts $c_1, c_2, ..., c_{6 - k}$ satisfy:

$$
c_1 + c_2 + \cdots + c_{6 - k} = 14 - 3k
$$

$$
c_i \geq 0
$$

- The number of solutions is given by the stars-and-bars theorem:

$$
\binom{(14 - 3k) + (6 - k) - 1}{(6 - k) - 1} = \binom{(14 - 3k) + (6 - k) - 1}{5 - k}
$$

Step 4: Count the Number of Sequences

- For a fixed set of counts, the number of sequences is:

$$
\frac{14!}{(3!)^{k} \prod_{i=1}^{6 - k} c_i!}
$$

- Summing over all possible distributions of counts c_i, we obtain the total number of sequences for each k.

Probabilistic Calculations

1. Probability of a Specific Configuration

- The probability of a particular configuration with counts $c_1, c_2, ..., c_6$, where exactly k faces appear exactly three times, is:

$$
P = \frac{14!}{\prod_{i=1}^{6} c_i!} \left(\frac{1}{6}\right)^{14}
$$

- Since the die is fair and rolls are independent, the multinomial probability applies.

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2. Expected Number of Faces Appearing Exactly Three Times

- By symmetry, each face has an equal chance of appearing exactly three times.

- The expected number of faces with exactly three occurrences is:

$$
E = 6 \times P(\text{a particular face appears exactly 3 times})
$$

Calculating this precisely involves summing over all configurations, which can be complex, but the expectation can be approximated or calculated using the properties of the multinomial distribution.

Practical Examples and Calculations

Let's illustrate the concepts with specific cases:

Example 1: Exactly 1 Face Appears Exactly Three Times

- Choose 1 face out of 6:

$$
\binom{6}{1} = 6
$$

- Remaining 11 rolls are distributed among 5 faces:

$$
c_1 + c_2 + c_3 + c_4 + c_5 = 11
$$

- Number of solutions:

$$
\binom{11 + 5 - 1}{5 - 1} = \binom{15}{4} = 1365
$$

- Number of sequences for each distribution:

$$
\frac{14!}{(3!) \times c_1! c_2! c_3! c_4! c_5!}
$$

- Summing over all solutions yields the total number of sequences where exactly one face appears three times.

Example 2: Exactly 2 Faces Appear Exactly Three Times

- Choose 2 faces:

$$
\binom{6}{2} = 15
$$

- Remaining 8 rolls among 4 faces:

$$
c_1 + c_2 + c_3 + c_4 = 8
$$

- Number of solutions:

$$
\binom{8 + 4 - 1}{4 - 1} = \binom{11}{3} = 165
$$

- For each, calculate the number of sequences and sum accordingly.

Application: Calculating Probabilities for Specific Outcomes

1. Probability that exactly one face appears exactly three times

- Sum over all configurations where one face appears three times and the remaining counts sum to 11 among the other faces.

- Multiply by the number of ways to choose the face and the counts, then divide by total possible sequences 6^{14}.

2. Probability that exactly two faces appear exactly three times each

- Similar calculation, considering the combinations of faces and distributions.

3. Extending to other values of k

- The process is analogous for $k=3$ or $k=4$, with adjustments for the sum of remaining counts.

Insights and Expected Outcomes

1. Expected Number of Faces Appearing Exactly Three Times

- Due to symmetry and the multinomial distribution, the expected number of faces appearing exactly three times in 14 rolls is approximately:

$$
E \approx
$$

Frequently Asked Questions

What is the probability that a specific face appears exactly three times when a fair die is rolled 14 times?

The probability that a specific face appears exactly three times in 14 rolls is given by the binomial probability: $C(14,3) (1/6)^3 (5/6)^{11}$.

How can we compute the expected number of faces that appear exactly three times in 14 rolls of a fair die?

Since each face has the same probability, the expected number of faces appearing exactly three times is 6 times the probability that a given face appears exactly three times, i.e., $6 C(14,3) (1/6)^3 (5/6)^{11}$.

What is the probability that exactly k faces appear exactly three times in 14 rolls of a fair die?

This involves summing over the possible combinations where k faces appear exactly three times, and the remaining faces appear in any way that sums to the remaining rolls, which is a complex multinomial problem typically approached with advanced combinatorial methods.

Which face is most likely to appear exactly three times in 14 rolls of a fair die?

All faces are equally likely due to symmetry; therefore, each specific face has the same probability of appearing exactly three times.

How does increasing the number of rolls affect the expected number of faces that appear exactly three times?

Increasing the number of rolls generally increases the expected number of faces appearing exactly three times, following the linearity of expectation based on the binomial probability.

Additional Resources

A Fair Die Is Rolled 14 Times. Let Be The Number Of Faces That Appear Exactly Three Times. Which Of The

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Introduction

Rolling dice is a classic problem in probability theory, often used to illustrate fundamental concepts such as discrete probability distributions, combinatorial calculations, and the law of large numbers. In this detailed analysis, we explore a specific scenario involving a fair six-sided die rolled 14 times. Our primary focus is on the random variable B, defined as the number of faces that appear exactly three times in these 14 rolls.

This problem combines elements of combinatorics, probability, and expectation calculations, making it a rich case study for understanding how discrete probability distributions behave, especially under conditions involving multiple outcomes and repeated trials. We will systematically examine the problem's components, derive formulas, and interpret the results, providing insights into the underlying structure and implications.

Setting Up the Problem

The Scenario

- Number of Rolls (n): 14
- Number of Faces (k): 6 (faces numbered 1 through 6)

- Fair Die: Each face has an equal probability $\frac{1}{6}$ of appearing on any roll
- Random Variable B: The number of faces that appear exactly three times in the 14 rolls

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Fundamental Concepts

Before delving into calculations, it’s crucial to understand some core concepts that underpin the problem:

- Multinomial Distribution: Since each roll is independent and has 6 possible outcomes with equal probability, the joint distribution of face counts follows a multinomial distribution.

- Indicator of Exactly Three Occurrences: For each face, the number of times it appears can be modeled as a binomial random variable, and the event that it appears exactly three times is a specific binomial probability.

- Counting Faces with Specific Frequencies: The random variable B counts how many faces satisfy a particular frequency condition (exactly three appearances). This involves combinatorial counting of subsets of faces and their occurrence counts.

Step 1: Distribution of Face Counts

In the context of rolling a fair die 14 times:

- Let X_i be the number of times face i appears for $i = 1, 2, \ldots, 6$.
- The vector $(X_1, X_2, \ldots, X_6)$ follows a multinomial distribution with parameters $n=14$ and probabilities $p_i = \frac{1}{6}$.

The joint probability mass function (pmf):

$$
P(X_1 = x_1, \ldots, X_6 = x_6) = \frac{14!}{x_1! x_2! \cdots x_6!} \left(\frac{1}{6}\right)^{x_1 + \cdots + x_6}
$$

subject to:

$$
x_1 + x_2 + \cdots + x_6 = 14
$$

and

$$
x_i \geq 0, \quad \forall i
$$

Step 2: Distribution of the Number of Faces Appearing Exactly Three Times

Our goal is to analyze the distribution of the discrete random variable:

$$
B = \text{Number of faces with } X_i = 3
$$

Given the symmetry among faces, the focus reduces to:

- For each face i, define the indicator variable:

$$
I_i =
\begin{cases}
1, & \text{if } X_i = 3 \\
0, & \text{otherwise}
\end{cases}
$$

- The random variable:

$$
B = \sum_{i=1}^6 I_i
$$

Step 3: Calculating the Distribution of B

Approach

Since the faces are exchangeable, the distribution of B hinges on:

- The probabilities that exactly j faces appear exactly three times, for $j = 0, 1, ..., 6$.

Expressed mathematically:

$$
P(B = j) = \text{Probability that exactly } j \text{ faces have } X_i = 3
$$

This involves combinatorial counting:

- Selecting which j faces are to have exactly three appearances.
- Assigning counts to the remaining faces ensuring the total sum is 14.
- Ensuring that the counts for the other faces are not exactly three (to prevent overcounting).

Step 4: Combinatorial Analysis

Selecting Faces with Exactly Three Occurrences

- Number of ways to choose j faces out of 6:

$$
\binom{6}{j}
$$

- For each such subset, assign counts as follows:

$$
x_{i} = 3, \quad \text{for } i \in \text{chosen subset}
$$

- The remaining $6 - j$ faces must sum to:

$$
14 - 3j
$$

- Each of these remaining faces can have counts $x_i \geq 0$, but not equal to 3.

Step 5: Distribution of Remaining Counts

This is the crux of the problem. For a fixed j, the counts for the remaining $6 - j$ faces:

$$
x_{i} \geq 0, \quad x_i \neq 3
$$

Sum to:

$$
S = 14 - 3j
$$

Note:

- The counts x_i are non-negative integers.
- The sum S must be at least 0 and at most 14.

Step 6: Calculating $P(B = j)$

The probability that exactly j faces appear exactly three times:

$$
P(B = j) = \binom{6}{j} \times \frac{(14)!}{(3)^j \prod_{i=1}^{6} x_i!} \times
$$

\left(\frac{1}{6}\right)^{14}
\]

but more precisely, the probability involves summing over all configurations where:

- Exactly \(j \) faces have counts equal to 3.
- The remaining counts sum to \(S = 14 - 3j \), with each count not equal to 3.

Due to the combinatorial complexity, the explicit probability \(P(B = j) \) can be computed via recursive enumeration or generating functions, but for practical purposes, we focus on the expectation and variance first.

Step 7: Expected Value of \(B \)

Using linearity of expectation:

\[
E[B] = \sum_{i=1}^{6} P(X_i = 3)
\]

Since the die rolls are independent and identical, and due to symmetry:

\[
E[B] = 6 \times P(X_i = 3)
\]

Calculating \(P(X_i = 3) \):

\[
P(X_i = 3) = \binom{14}{3} \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^{11}
\]

where:

- \(\binom{14}{3} \) accounts for choosing the 3 positions where face \(i \) appears.

Numerically:

\[
\binom{14}{3} = \frac{14 \times 13 \times 12}{3 \times 2 \times 1} = 364
\]

and

\[
P(X_i = 3) = 364 \times \left(\frac{1}{6}\right)^3 \times \left(\frac{5}{6}\right)^{11}
\]

Calculating:

$$
\left(\frac{1}{6}\right)^3 = \frac{1}{216}
$$

$$
\left(\frac{5}{6}\right)^{11} \approx 0.1615
$$

Thus:

$$
P(X_i = 3) \approx 364 \times \frac{1}{216} \times 0.1615 \approx 364 \times 0.00463 \approx 1.687
$$

But since probabilities are less than 1, this indicates the need to recalculate carefully:

$$
P(X_i=3) = \binom{14}{3} \times \left(\frac{1}{6}\right)^3 \times \left(\frac{5}{6}\right)^{11}
$$

Numerical approximation:

$$
\binom{14}{3} = 364
$$

$$
\left(\frac{1}{6}\right)^3 = \frac{1}{216} \approx 0.0046296
$$

$$
\left(\frac{5}{6}\right)^{11} \approx e^{11 \times \ln(5/6)} \approx e^{11 \times -0.182} \approx e^{-2.002} \approx 0.135
$$

Now:

$$
P(X_i=3) \approx 364 \times 0.0046296 \times 0.135 \approx 364 \times 0.000624 \approx 0.227
$$

Therefore,

$$
E[B] = 6 \times 0.227 \
$$

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