A Random Variable X Has The Pdf Shown Below: F X(x)={ Cx(1x 2)00x1elsewhere A. Find C And Plot The Pdf.

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Understanding probability density functions (pdfs) is fundamental in the study of probability and statistics. When dealing with continuous random variables, the pdf provides the likelihood of the variable taking on a specific value within a range. In this article, we will analyze a given pdf, determine the constant C, and illustrate the shape of the function through plotting. We will also explore the concepts involved in deriving the constant, verifying the validity of the pdf, and visualizing the distribution for better comprehension.

Introduction to Probability Density Functions (pdfs)

Before diving into the problem, it's essential to review what a probability density function entails:

- Definition: A pdf $f_X(x)$ describes the likelihood of a continuous random variable X taking on a value near x. Unlike probability mass functions for discrete variables, pdfs are used for continuous variables.

- Properties of a pdf:
1. Non-negativity: $f_X(x) \geq 0$ for all x.
2. Total probability: The integral over the entire support equals 1:
$$
\int_{-\infty}^\infty f_X(x) \, dx = 1
$$

- Cumulative Distribution Function (CDF): The CDF $F_X(x)$ is the integral of the pdf from $-\infty$ up to x:
$$
F_X(x) = \int_{-\infty}^x f_X(t) \, dt
$$

In our problem, the given function appears to be a piecewise function, which is common when defining pdfs over specific intervals.

Understanding the Given Pdf: $F_X(x)$

The problem states:

> A Random Variable X has the pdf shown below:
>
> $$
> F_X(x) =
> \begin{cases}
> Cx(1 - x^2), & x \text{ in a certain interval} \\
> 0, & \text{elsewhere}
> \end{cases}
> $$

(Note: The original prompt seems to have some typographical issues, but based on context, it's likely that $F_X(x)$ was intended to be $f_X(x)$, the pdf, rather than the CDF. Alternatively, it could be referring to the CDF, but typically, the notation $F_X(x)$ is used for the CDF. For clarity, we'll assume it's the pdf, $f_X(x)$.)

Assumption: The function is a piecewise function defined over a specific interval, say $x \in [a, b]$, with $f_X(x) = C \cdot x(1 - x^2)$ within that interval, and zero elsewhere.

The key points are:

- $f_X(x)$ is non-zero only over a specific range.
- The function involves a constant C which we need to determine.
- The function involves polynomial terms, which are common in pdfs.

Note: For the purpose of this analysis, we will assume the interval is $0 \leq x \leq 1$, a common interval for such functions, unless specified otherwise.

Step 1: Clarify the Support and Function Form

Given the typical structure, the pdf might be:

$$
f_X(x) =
\begin{cases}
C \cdot x(1 - x^2), & 0 \leq x \leq 1 \\
0, & \text{elsewhere}
\end{cases}
$$

This assumption aligns with standard practice, but if the original problem specifies different support, adjust accordingly.

Step 2: Find the Constant C

To find C, we utilize the fundamental property of pdfs:

$$
\int_{-\infty}^\infty f_X(x) \, dx = 1
$$

Given the support, this reduces to:

$$
\int_{0}^{1} C \cdot x(1 - x^2) \, dx = 1
$$

$$
C \int_0^1 x(1 - x^2) \, dx = 1
$$

Let's compute this integral step-by-step.

Calculating the Integral

$$
\int_0^1 x(1 - x^2) \, dx
$$

Expand the integrand:

$$
x(1 - x^2) = x - x^3
$$

Thus,

$$
\int_0^1 (x - x^3) \, dx = \int_0^1 x \, dx - \int_0^1 x^3 \, dx
$$

Compute each integral:

- $\int_0^1 x \, dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$

- $\int_0^1 x^3 \, dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4}$

Putting it together:

$$
\int_0^1 (x - x^3) \, dx = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}
$$

Recall the original integral with C:

$$
C \times \frac{1}{4} = 1
$$

Solve for C:

$$
C = 4
$$

Result: The constant C is 4.

Step 3: Write the Final Pdf Expression

The pdf of the random variable X:

$$
f_X(x) =
\begin{cases}
4 x (1 - x^2), & 0 \leq x \leq 1 \\
0, & \text{elsewhere}
\end{cases}
$$

This function is valid as it satisfies the properties of a pdf.

Step 4: Verify the Pdf

To ensure correctness, verify:

$$
\int_{-\infty}^\infty f_X(x) \, dx = 1
$$

which we've already confirmed through the calculation of C. Additionally, check that $f_X(x) \geq 0$ over its support:

- For $x \in [0, 1]$, $x \geq 0$ and $1 - x^2 \geq 0$ for $x \leq 1$, so $f_X(x) \geq 0$.

Step 5: Plotting the Pdf

Visualizing the pdf helps in understanding the distribution's shape.

Method:

- Use graphing tools like Desmos, GeoGebra, or programming languages such as Python with matplotlib.
- Plot $f_X(x)$ over the interval $[0, 1]$.

Expected Shape:

- At $x = 0$: $f_X(0) = 0$.
- At $x = 1$: $f_X(1) = 4 \times 1 \times (1 - 1) = 0$.
- The function reaches a maximum somewhere in $(0, 1)$.

Analytical Max Point:

Find the maximum by taking the derivative:

$$
f_X(x) = 4 x (1 - x^2) = 4x - 4x^3
$$

Derivative:

$$
f_X'(x) = 4 - 12 x^2
$$

Set $f_X'(x) = 0$:

$$
4 - 12 x^2 = 0 \Rightarrow 12 x^2 = 4 \Rightarrow x^2 = \frac{1}{3} \Rightarrow x = \pm \frac{1}{\sqrt{3}}
$$

Since the support is $[0, 1]$, consider $x = \frac{1}{\sqrt{3}} \approx 0.577$.

Calculate f_X at this point:

$$
f_X\left(\frac{1}{\sqrt{3}}\right) = 4 \times \frac{1}{\sqrt{3}} \times \left(1 - \frac{1}{3}\right) = 4 \times \frac{1}{\sqrt{3}} \times \frac{2}{3} = \frac{8}{3} \times \frac{1}{\sqrt{3}} = \frac{8}{3 \sqrt{3}}
$$

Numerically:

$$
f_X\left(\frac{1}{\sqrt{3}}\right) \approx \frac{8}{3 \times 1.732} \approx \frac{8}{5.196} \approx 1.539
$$

Thus, the maximum pdf value is approximately 1.539 at

Frequently Asked Questions

What is the given probability density function (PDF) for the random variable X?

The PDF is given as $f_X(x) = Cx(1 - x)$ for $0 \leq x \leq 1$, and 0 elsewhere.

How do we determine the value of the constant C in the PDF?

We find C by ensuring the total probability integrates to 1: ∫ from 0 to 1 of $Cx(1 - x)\,dx = 1$.

What is the integral to compute C for the given PDF?

The integral is $C \int_0^1 x(1 - x)\,dx = 1$.

How do you evaluate the integral $\int_0^1 x(1 - x)\,dx$?

Expand the integrand: $x - x^2$, then integrate term by term: $(x^2/2) - (x^3/3)$ evaluated from 0 to 1.

What is the value of the integral $\int_0^1 x(1 - x)\,dx$?

Evaluating gives $(1/2) - (1/3) = (3/6) - (2/6) = 1/6$.

What is the value of the constant C in the PDF?

Since $C\,(1/6) = 1$, $C = 6$.

How do you plot the PDF $f_X(x) = 6x(1 - x)$ for x in [0, 1]?

Plot the parabola $y = 6x(1 - x)$ over the interval [0,1], noting that it peaks at $x=0.5$ and is zero at the endpoints.

What are the key features of the PDF $f_X(x) = 6x(1 - x)$?

It is a symmetric parabola with maximum at $x=0.5$ and zero at $x=0$ and $x=1$.

How can you verify that the PDF integrates to 1 after finding C?

By integrating $f_X(x) = 6x(1 - x)$ from 0 to 1 and confirming the total area equals 1.

What do the parameters in the PDF tell us about the distribution of X?

The distribution is symmetric with a peak at x=0.5, indicating X is most likely near the center of the interval.

Additional Resources

Understanding the Probability Density Function and Its Normalization: A Deep Dive into the Random Variable X

Introduction

In the realm of probability theory and statistics, the probability density function (pdf) serves as a fundamental building block for understanding the behavior of continuous random variables. When dealing with a random variable X, the pdf describes how the probability mass is distributed over the possible values of X. In this article, we explore a particular problem involving a given pdf of the form:

$$
f_X(x) = \begin{cases}
C \cdot x (1 - x)^2 & \text{for } 0 \leq x \leq 1 \\
0 & \text{elsewhere}
\end{cases}
$$

The goal is to determine the constant C ensuring the pdf’s validity, plot the function, and analyze its properties. This comprehensive exploration will illuminate the process of normalization, the interpretation of the pdf, and the graphical representation, providing valuable insights for students, educators, and practitioners alike.

Understanding the Probability Density Function (pdf)

What Is a Probability Density Function?

A probability density function (pdf) for a continuous random variable X is a non-negative function that integrates to 1 over its domain:

$$
\int_{-\infty}^\infty f_X(x) \, dx = 1
$$

This integral represents the total probability, which must be 1 for any valid pdf. The function $f_X(x)$ does not give probabilities directly but rather densities—probabilities over an interval are obtained by integrating the pdf over that interval.

Characteristics of the Given pdf

The provided pdf:

$$
f_X(x) = C \cdot x (1 - x)^2, \quad 0 \leq x \leq 1
$$

is a polynomial function defined over the unit interval [0, 1]. Its shape is dictated by the factors x and $(1 - x)^2$, which influence the skewness and modality of the distribution.

Determining the Normalizing Constant C

The Necessity of Normalization

To validate $f_X(x)$ as a true pdf, we must ensure:

$$
\int_{-\infty}^\infty f_X(x) \, dx = 1
$$

Given that $f_X(x) = 0$ outside [0, 1], this reduces to:

$$
\int_0^1 C \cdot x (1 - x)^2 \, dx = 1
$$

Our task is to solve for C.

Computing the Integral

The integral becomes:

$$
C \int_0^1 x (1 - x)^2 \, dx = 1
$$

Let’s compute the integral:

$$
\int_0^1 x (1 - x)^2 \, dx
$$

Step-by-step calculation:

1. Expand $(1 - x)^2$:

\[

$$
(1 - x)^2 = 1 - 2x + x^2
$$

2. Express the integrand:

$$
x (1 - 2x + x^2) = x - 2x^2 + x^3
$$

3. Integrate term-by-term:

$$
\int_0^1 x \, dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}
$$

$$
\int_0^1 -2x^2 \, dx = -2 \left[\frac{x^3}{3}\right]_0^1 = -2 \times \frac{1}{3} = -\frac{2}{3}
$$

$$
\int_0^1 x^3 \, dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4}
$$

4. Sum the results:

$$
\frac{1}{2} - \frac{2}{3} + \frac{1}{4}
$$

Express all fractions with denominator 12:

$$
\frac{6}{12} - \frac{8}{12} + \frac{3}{12} = \frac{6 - 8 + 3}{12} = \frac{1}{12}
$$

Therefore:

$$
\int_0^1 x (1 - x)^2 \, dx = \frac{1}{12}
$$

Solving for C

Recall:

$$
C \times \frac{1}{12} = 1 \Rightarrow C = 12
$$

Result:

$$
\boxed{
C = 12
}
$$

This normalization constant ensures that the total probability integrates to 1, confirming that $f_X(x)$ is a valid pdf.

Plotting the Probability Density Function

Visualizing the Function

With C determined, the pdf simplifies to:

$$
f_X(x) = 12 \cdot x (1 - x)^2, \quad 0 \leq x \leq 1
$$

Key features:

- Shape: The function starts at zero at $x=0$, rises to a peak, and then declines back to zero at $x=1$.
- Skewness: The polynomial form suggests a skewed distribution, with a mode somewhere in (0,1).
- Maximum Point: Identified by setting the derivative to zero.

Analytical Determination of the Mode

To find the mode (the value of x at which the pdf attains its maximum), differentiate $f_X(x)$:

$$
f_X(x) = 12x(1 - x)^2
$$

Derivative:

$$
f'_X(x) = 12 \left[(1 - x)^2 + x \times 2(1 - x)(-1) \right]
$$

Simplify:

$$

$$
f'_X(x) = 12 \left[(1 - x)^2 - 2x (1 - x) \right]
$$

Factor:

$$
f'_X(x) = 12 (1 - x) \left[(1 - x) - 2x \right]
$$

$$
f'_X(x) = 12 (1 - x) (1 - x - 2x) = 12 (1 - x) (1 - 3x)
$$

Set $f'_X(x) = 0$:

$$
12 (1 - x)(1 - 3x) = 0
$$

Solutions:

$$
x = 1 \quad \text{or} \quad x = \frac{1}{3}
$$

Since $f_X(1) = 0$, the maximum occurs at $x = \frac{1}{3}$.

Compute $f_X(1/3)$:

$$
f_X\left(\frac{1}{3}\right) = 12 \times \frac{1}{3} \times \left(1 - \frac{1}{3}\right)^2 = 12 \times \frac{1}{3} \times \left(\frac{2}{3}\right)^2
$$

$$
= 12 \times \frac{1}{3} \times \frac{4}{9} = 4 \times \frac{4}{9} = \frac{16}{9} \approx 1.7778
$$

This confirms the peak of the pdf at $x = \frac{1}{3}$.

Graphical Representation

To visualize the pdf, plotting over the interval [0,1]:

- The function starts at 0 at $x=0$.
- Rises to its maximum at $x=1/3$, approximately 1.78.
- Falls back to 0 at $x=1$.

Such a graph illustrates the skewed, unimodal distribution with a clear peak, characteristic of Beta-like distributions.

Analytical Significance and Applications

Distribution Type

The form $f_X(x) = C x (1 - x)^2$ resembles the Beta distribution's pdf:

$$
f_{\text{Beta}}(x; \alpha, \beta) = \frac{x^{\alpha - 1}(1 - x)^{\beta - 1}}{B(\alpha, \beta)}
$$

where $B(\alpha, \beta)$ is the Beta function.

Matching parameters:

$$
\alpha - 1 = 1 \Rightarrow \alpha = 2
$$
$$
\beta - 1 = 2 \Rightarrow \beta = 3
$$

and

$$
C = \frac{1}{B(2,3)} = \frac{1}{\frac{\Gamma(2) \Gamma(3)}{\Gamma(5)}} = \frac{1}{\frac{1! \times 2!}{4!}} = \frac{1}{\frac{1 \times 2}{24}} = \frac{1}{\frac{2}{24}} = \frac{24}{2} = 12
$$

which aligns with our previous calculation

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