A Spherical Balloon Is Inflated With Helium At A Rate Of 140pift^3/min How Fast Is The Balloon's Radius

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A spherical balloon is inflated with helium at a rate of 140 cubic feet per minute. How fast is the balloon's radius increasing? This question involves understanding the relationship between the volume of a sphere and its radius, and how the rate of change of volume relates to the rate of change of the radius over time. In this comprehensive guide, we will explore the mathematical principles behind this problem, step through the calculations step-by-step, and provide insights into related concepts such as related rates in calculus, unit conversions, and practical applications.

Understanding the Problem: The Basics

What Is Being Asked?

The problem asks for the rate at which the radius of a spherical helium balloon is increasing, given the rate at which its volume is increasing. More specifically:
- Volume rate of change: 140 ft^3/min
- Find: Rate of change of radius (dr/dt)

Key Concepts Involved

- Volume of a sphere: $V = \frac{4}{3} \pi r^3$
- Derivative of volume with respect to time: $\frac{dV}{dt}$
- Rate of change of the radius: $\frac{dr}{dt}$

Mathematical Foundations

The Volume Formula for a Sphere

The volume V of a sphere with radius r is given by:

$$
V = \frac{4}{3} \pi r^3
$$

\]

This formula establishes a relationship between the volume and the radius.

Differentiating the Volume with Respect to Time

To connect the change in volume (dV/dt) with change in radius (dr/dt), differentiate both sides of the volume formula with respect to time t:

$$
\frac{dV}{dt} = 4 \pi r^2 \frac{dr}{dt}
$$

This is derived using the chain rule, considering that V is a function of r, which in turn is a function of t.

Expressing the Rate of Change of the Radius

Rearranging the differential equation:

$$
\frac{dr}{dt} = \frac{1}{4 \pi r^2} \frac{dV}{dt}
$$

This formula allows us to calculate the rate at which the radius is increasing, provided we know the current radius r and the rate of volume increase dV/dt.

Calculating the Rate of Change of the Radius

Given Data

- $\frac{dV}{dt} = 140 \text{ ft}^3/\text{min}$
- Initial radius r (unknown; needs to be determined or considered at a specific volume)

Step 1: Determine the Radius at a Given Volume

Since the rate of volume change is given, but not the current volume or radius, we need to understand how the radius varies as the volume increases. For the rate calculation at a specific moment, we must know r.

Suppose we are asked to find the rate of radius change at a specific volume V, then:

$$
r = \left(\frac{3V}{4\pi} \right)^{1/3}
$$

Alternatively, if the problem is to find how fast the radius is increasing when the volume is known, plug in the current volume to get r.

Step 2: Express dr/dt in terms of V

Using the volume formula:

$$
r = \left(\frac{3V}{4 \pi} \right)^{1/3}
$$

Then:

$$
r^2 = \left(\frac{3V}{4 \pi} \right)^{2/3}
$$

And the rate of change:

$$
\frac{dr}{dt} = \frac{1}{4 \pi r^2} \times \frac{dV}{dt}
$$

Substituting r:

$$
\frac{dr}{dt} = \frac{1}{4 \pi \left(\frac{3V}{4 \pi} \right)^{2/3}} \times 140
$$

Simplifying:

$$
\frac{dr}{dt} = \frac{140}{4 \pi} \times \left(\frac{4 \pi}{3V} \right)^{2/3}
$$

This formula can be used to compute dr/dt at any specific volume V.

Practical Calculation: An Example

Suppose the balloon has a current volume of 100 ft³. Let's find the rate at which the radius is increasing at this volume.

Step 1: Find the current radius r

$$
r = \left(\frac{3 \times 100}{4 \pi} \right)^{1/3} \approx \left(\frac{300}{4 \pi} \right)^{1/3}

Calculating:

$$
\frac{300}{4 \pi} \approx \frac{300}{12.566} \approx 23.87
$$

Thus,

$$
r \approx (23.87)^{1/3} \approx 2.89 \text{ ft}
$$

Step 2: Calculate dr/dt

Using the earlier formula:

$$
\frac{dr}{dt} = \frac{1}{4 \pi r^2} \times \frac{dV}{dt}
$$

Substitute $r \approx 2.89$ ft:

$$
r^2 \approx (2.89)^2 \approx 8.35
$$

Then:

$$
\frac{dr}{dt} \approx \frac{1}{4 \pi \times 8.35} \times 140
$$

Calculate denominator:

$$
4 \pi \times 8.35 \approx 12.566 \times 8.35 \approx 104.86
$$

Finally:

$$
\frac{dr}{dt} \approx \frac{140}{104.86} \approx 1.34 \text{ ft/min}
$$

Answer: The radius of the balloon is increasing at approximately 1.34 feet per minute when the volume is 100 ft^3.

Additional Considerations and Real-World Applications

Impact of the Rate of Inflation

Understanding how quickly a balloon's radius increases is essential in various fields:
- Aerospace Engineering: Designing balloons or airships with controlled inflation rates.
- Event Planning: Ensuring balloons are inflated safely without bursting.
- Physics Education: Demonstrating related rates concepts in calculus.

Unit Conversions and Accuracy

While the example uses feet and cubic feet, similar calculations can be performed in meters and cubic meters. Always ensure consistency in units to avoid errors.

Limitations and Assumptions

- The calculations assume the balloon remains perfectly spherical during inflation.
- Helium's behavior is ideal; real-world factors like pressure and temperature changes are ignored.
- The rate of inflation is constant; in practice, it may vary.

Summary of Steps to Find the Rate of Radius Change

1. Identify the volume formula for the sphere: $V = \frac{4}{3} \pi r^3$.
2. Differentiate with respect to time to relate dV/dt and dr/dt: $\frac{dV}{dt} = 4 \pi r^2 \frac{dr}{dt}$.
3. Rearranged to find dr/dt: $\frac{dr}{dt} = \frac{1}{4 \pi r^2} \times \frac{dV}{dt}$.
4. Determine the current radius based on the current volume: $r = \left(\frac{3V}{4 \pi} \right)^{1/3}$.
5. Calculate dr/dt using the known dV/dt and the current radius.

Conclusion

Understanding how a sphere's radius changes as it is inflated at a known volumetric rate involves fundamental calculus principles like related rates. By applying the volume formula for a sphere and differentiating with respect to time, we derive a straightforward relationship that allows us to compute the rate of change of the radius at any moment. Whether for educational purposes, engineering design, or practical balloon inflation, these calculations are vital for precise and safe operations. Remember, always account for units and real-world factors when applying these mathematical concepts to actual scenarios.

Frequently Asked Questions

What is the relationship between the volume of a spherical balloon and its radius?

The volume V of a spherical balloon is related to its radius r by the formula $V = (4/3)\pi r^3$.

If a balloon is being inflated at a rate of 140 ft^3/min, how can we find the rate at which the radius is increasing?

We differentiate the volume formula with respect to time to relate dV/dt to dr/dt: $dV/dt = 4\pi r^2\ dr/dt$, then solve for $dr/dt = (dV/dt) / (4\pi r^2)$.

What is the formula to calculate the rate of change of the radius of the balloon given the rate of volume increase?

The rate of change of radius is $dr/dt = (dV/dt) / (4\pi r^2)$.

How does the radius of the balloon change over time if the volume increases at 140 ft^3/min?

The radius increases at a rate given by $dr/dt = 140 / (4\pi r^2)$ ft/min, which depends on the current radius r.

What factors influence how fast the radius of the helium balloon is expanding?

The expansion rate depends on the volume inflow rate (140 ft^3/min) and the current radius of the balloon.

If the balloon's radius is 5 ft, how fast is the radius increasing at that moment?

At r = 5 ft, $dr/dt = 140 / (4\pi\ 5^2) = 140 / (4\pi\ 25) = 140 / (100\pi) \approx 0.445$ ft/min.

How does the radius's rate of increase change as the balloon gets larger?

As the radius increases, the denominator $4\pi r^2$ grows, so dr/dt decreases, meaning the radius increases more slowly over time.

What practical applications require understanding how quickly a helium balloon's radius expands?

Applications include designing balloons for events, calculating gas flow rates in manufacturing, and ensuring safety in pressurized systems.

Can we determine the exact rate of radius change at any moment without knowing the current radius?

No, because dr/dt depends on the current radius r; without r, we cannot compute the exact rate at that moment.

Additional Resources

A Spherical Balloon Is Inflated With Helium At A Rate Of 140 ft^3/min How Fast Is The Balloon's Radius?

Imagine you're filling a perfectly round balloon with helium, watching it grow and expand. Have you ever wondered how fast the radius of that spherical balloon increases as it's being inflated? This question is not just a playful curiosity; it's a fascinating application of calculus, specifically related to related rates — a branch of differential calculus that deals with how the rate of change of one quantity affects the rate of change of another. In this guide, we'll explore how to determine how fast the radius of a spherical balloon is increasing when helium is being pumped in at a known rate.

Let's delve into the physics and mathematics behind this problem, understand the key concepts, and develop a step-by-step solution that can be applied to similar real-world problems.

Understanding the Problem

Before diving into calculations, it's essential to understand the problem setup clearly:

- Given Data:
- The rate at which helium is being pumped into the balloon: 140 ft^3/min
- The shape of the balloon: sphere

- Goal:
- Find the rate of change of the radius of the balloon with respect to time, i.e., dr/dt, at a specific instant or generally as a function of time.

This problem is a classic example of a related rates problem. It involves connecting the volume of a sphere to its radius, then differentiating to find how the radius changes over time as the volume increases.

Fundamental Concepts and Formulas

1. Volume of a Sphere

The volume V of a sphere of radius r is given by:

$$ V = \frac{4}{3} \pi r^3 $$

2. Derivatives and Related Rates

Since volume V and radius r are related through the above formula, differentiating both sides with respect to time t gives:

$$ \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} $$

This is the key relation that links the rate of change of volume to the rate of change of radius.

Step-by-Step Solution Approach

Step 1: Write Down Known Quantities

- Rate of volume increase: $\frac{dV}{dt} = 140\, \text{ft}^3/\text{min}$
- Radius at the instant: r (unknown, but can be expressed or evaluated at specific points if needed)

Step 2: Express $\frac{dr}{dt}$

Rearranged from the derivative relation:

$$
\frac{dr}{dt} = \frac{1}{4\pi r^2} \frac{dV}{dt}
$$

Step 3: Plug in Known Values

To find how fast the radius is increasing, substitute $\frac{dV}{dt} = 140$ ft^3/min:

$$
\frac{dr}{dt} = \frac{140}{4\pi r^2}
$$

Note that $\frac{dr}{dt}$ depends on the current radius r. To evaluate a numerical value, you'll need to know r at the specific moment.

How to Find the Radius at a Given Moment

Suppose you want to find how fast the radius is increasing when the radius is a certain value, say r = 10 ft. Then:

$$
\frac{dr}{dt} = \frac{140}{4\pi (10)^2} = \frac{140}{4\pi \times 100} = \frac{140}{400\pi} = \frac{7}{20\pi} \text{ ft/min}
$$

Calculating numerically:

$$
\frac{7}{20\pi} \approx \frac{7}{62.8319} \approx 0.1114 \text{ ft/min}
$$

Interpretation: When the radius is 10 ft, the radius is increasing at approximately 0.1114 ft/min.

Extending the Analysis: How the Radius Changes Over Time

1. Find the Radius as a Function of Time

Suppose you want to know $r(t)$, the radius at any time t. Start from the volume formula:

$$
V = \frac{4}{3} \pi r^3
$$

Since dV/dt is constant at 140 ft^3/min, integrate or model the volume over time:

$$
V(t) = V_0 + \left(\frac{dV}{dt}\right) t
$$

where V_0 is the initial volume. For simplicity, assume the balloon starts empty:

$$
V(t) = 140 t
$$

Now, express $r(t)$:

$$
V(t) = \frac{4}{3} \pi r(t)^3
$$

$$
r(t)^3 = \frac{3 V(t)}{4 \pi} = \frac{3 \times 140 t}{4 \pi} = \frac{420 t}{4 \pi}
$$

$$
r(t)^3 = \frac{105 t}{\pi}
$$

Thus:

$$
r(t) = \left(\frac{105 t}{\pi}\right)^{1/3}
$$

2. Find $\frac{dr}{dt}$ as a Function of Time

Differentiate r(t):

$$
r(t) = \left(\frac{105 t}{\pi}\right)^{1/3}
$$

Using the chain rule:

$$
\frac{dr}{dt} = \frac{1}{3} \left(\frac{105 t}{\pi}\right)^{-2/3} \times \frac{105}{\pi}
$$

Simplify:

$$
\frac{dr}{dt} = \frac{1}{3} \times \frac{105}{\pi} \times \left(\frac{105 t}{\pi}\right)^{-2/3}
$$

Or:

$$
\frac{dr}{dt} = \frac{35}{\pi} \times \left(\frac{105 t}{\pi}\right)^{-2/3}
$$

This function gives the rate at which the radius increases at any moment t.

Practical Considerations and Real-World Applications

Understanding how fast the radius of a balloon increases has many practical applications:

- Manufacturing and Quality Control: Ensuring balloons don't expand too rapidly, risking rupture.
- Aerospace Engineering: Designing weather balloons or other spherical containers.
- Physics and Chemistry: Modeling gas expansion in closed systems.
- Education: Demonstrating related rates problems to students.

Limitations and Assumptions

While this mathematical analysis provides a solid framework, real-world scenarios involve additional factors:

- Air Resistance and External Pressure: These can affect the expansion rate.

- Material Elasticity: The balloon's material might stretch non-linearly.
- Helium Leakages: Over time, helium may escape, affecting volume and rate calculations.
- Constant Rate Assumption: The analysis assumes a constant inflow rate, which might vary in practice.

Summary and Key Takeaways

- The rate of change of the radius of a spherical balloon $\frac{dr}{dt}$ can be found using the relation:

$$
\boxed{
\frac{dr}{dt} = \frac{1}{4\pi r^2} \times \frac{dV}{dt}
}
$$

- When helium is pumped in at 140 ft^3/min, the radius increases faster when the balloon is smaller and slows down as the radius grows larger.
- The radius as a function of time:

$$
r(t) = \left(\frac{105 t}{\pi}\right)^{1/3}
$$

and the rate of radius increase at any time:

$$
\frac{dr}{dt} = \frac{35}{\pi} \times \left(\frac{105 t}{\pi}\right)^{-2/3}
$$

- This problem exemplifies the power of calculus in solving real-world related rates questions, enabling us to predict and analyze dynamic changes in physical systems.

Final Thoughts

Whether you're filling a balloon or designing complex spherical structures, understanding how the radius changes over time based on inflow rates is crucial. By mastering related rates problems like this, you gain valuable insights into the interplay between geometry, calculus, and physical phenomena. Keep practicing with different shapes, rates, and initial conditions to deepen your understanding and application skills!

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