A Unit Mass Hangs In Equilibrium From A Spring With Constant = 1/16. Starting At = 0, A Force () = 3sin

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Understanding the dynamics of a mass attached to a spring under external forces is fundamental in physics and engineering. In this article, we explore the behavior of a unit mass hanging in equilibrium from a spring with a spring constant of $\frac{1}{16}$, subjected to a sinusoidal external force $F(t) = 3 \sin \omega t$, starting at time $t=0$. We delve into the mathematical modeling, analyze the solution of the resulting differential equation, and interpret the physical implications of the system's response.

Modeling the Physical System

System Description

- A unit mass (mass $m = 1$ kg) is attached vertically to a spring.
- The spring has a spring constant $k = \frac{1}{16}$ N/m.
- The mass is initially at equilibrium position $x=0$ when no external force is applied.
- An external sinusoidal force $F(t) = 3 \sin \omega t$ acts on the mass, with angular frequency ω.

Assumptions and Simplifications

- The system is mass-spring-damper free of damping.
- The motion is vertical and only occurs in one dimension.
- The initial conditions are:
- Displacement $x(0) = 0$.
- Velocity $v(0) = 0$.

Mathematical Formulation

Deriving the Differential Equation

Applying Newton's second law, the sum of forces is:

$$
m \frac{d^2 x}{dt^2} = -k x + F(t)
$$

\]

Since $m=1$, $k= \frac{1}{16}$, and $F(t) = 3 \sin \omega t$, the equation simplifies to:

$$
\frac{d^2 x}{dt^2} + \frac{1}{16} x = 3 \sin \omega t
$$

This is a nonhomogeneous second-order linear differential equation.

Standard Form

Expressed explicitly:

$$
x''(t) + \frac{1}{16} x(t) = 3 \sin \omega t
$$

where $x''(t) = \frac{d^2 x}{dt^2}$.

Solving the Differential Equation

Homogeneous Equation

Solve:

$$
x'' + \frac{1}{16} x = 0
$$

Characteristic equation:

$$
r^2 + \frac{1}{16} = 0 \Rightarrow r = \pm i \frac{1}{4}
$$

Solution:

$$
x_h(t) = C_1 \cos \frac{t}{4} + C_2 \sin \frac{t}{4}
$$

where C_1, C_2 are determined by initial conditions.

Particular Solution

Since the forcing function is sinusoidal $3 \sin \omega t$, we look for a particular solution of the form:

$$
x_p(t) = A \sin \omega t + B \cos \omega t
$$

Plug into the differential equation:

$$
x_p'' + \frac{1}{16} x_p = 3 \sin \omega t
$$

Calculating derivatives:

$$
x_p'' = -A \omega^2 \sin \omega t - B \omega^2 \cos \omega t
$$

Substituting:

$$
(-A \omega^2 \sin \omega t - B \omega^2 \cos \omega t) + \frac{1}{16}(A \sin \omega t + B \cos \omega t) = 3 \sin \omega t
$$

Grouping terms:

$$
\left(-A \omega^2 + \frac{A}{16} \right) \sin \omega t + \left(- B \omega^2 + \frac{B}{16} \right) \cos \omega t = 3 \sin \omega t
$$

Matching coefficients:

$$
\begin{cases}
-A \omega^2 + \frac{A}{16} = 3 \\
- B \omega^2 + \frac{B}{16} = 0
\end{cases}
$$

Solve for A and B:

1. For B:

$$
(- \omega^2 + \frac{1}{16}) B = 0
$$

- If $B \neq 0$, then

$$
\omega^2 = \frac{1}{16}
$$

- Otherwise, $B=0$.

2. For A:

$$
A \left(- \omega^2 + \frac{1}{16} \right) = 3
$$

- If $\omega^2 \neq \frac{1}{16}$, then

$$
A = \frac{3}{ - \omega^2 + \frac{1}{16} }
$$

- If $\omega^2 = \frac{1}{16}$, resonance occurs, and a different approach is needed.

Case Analysis Based on ω

- Case 1: $\omega^2 \neq \frac{1}{16}$

$$
A = \frac{3}{ \frac{1}{16} - \omega^2 }
$$
$$
B=0
$$

- Case 2: $\omega^2 = \frac{1}{16}$

Resonance occurs; the particular solution must be modified, often involving t multiplying the sinusoid.

Complete Solution and Initial Conditions

The general solution:

$$
x(t) = x_h(t) + x_p(t)
$$

$$
x(t) = C_1 \cos \frac{t}{4} + C_2 \sin \frac{t}{4} + A \sin \omega t + B \cos \omega t
$$

Apply initial conditions:

$$
x(0) = 0 \Rightarrow C_1 + B = 0
$$
$$
x'(0) = 0 \Rightarrow - \frac{1}{4} C_1 + \frac{1}{4} C_2 + A \omega \cos 0 - B \omega \sin 0 = 0
$$

which simplifies to:

$$
- \frac{1}{4} C_1 + \frac{1}{4} C_2 + A \omega = 0
$$

Solve for C_1, C_2:

$$
C_1 = - B
$$
$$
- \frac{1}{4} (- B) + \frac{1}{4} C_2 + A \omega = 0
$$
$$
\frac{1}{4} B + \frac{1}{4} C_2 + A \omega = 0
$$

Express C_2:

$$
C_2 = - B - 4 A \omega
$$

Physical Interpretation of the Response

Steady-State Behavior

- The particular solution $x_p(t)$ dominates as $t \to \infty$.
- The system exhibits oscillations driven by the external force, with amplitude depending on ω.

Resonance Phenomenon

- When ω^2 approaches $\frac{1}{16}$, the denominator $\frac{1}{16} - \omega^2$ becomes small.
- This causes the amplitude A to grow large, indicating resonance.
- Practical systems require damping to prevent unbounded growth.

Amplitude of Steady-State Oscillations

- The amplitude of the oscillations induced by the forcing term:

$$
|A| = \left| \frac{3}{ \frac{1}{16} - \omega^2 } \right|
$$

- The response magnitude is highest near the natural frequency $\omega_0 = \frac{1}{4}$.

Implications and Applications

Engineering Design Considerations

- Avoiding resonance by choosing operational frequencies away from the system’s natural frequency.
- Incorporating damping mechanisms to limit oscillation amplitudes.
- Monitoring external forces that can induce large oscillations, potentially causing structural failure.

Real-World Examples

- Suspension bridges subjected to periodic loads.
- Building structures experiencing rhythmic forces like wind or seismic activity.
- Mechanical components in machinery where external vibrations are common.

Summary of Key Points

1. The system is modeled by a second-order linear differential equation with sinusoidal forcing.

2. The homogeneous solution captures natural oscillations, while the particular

Frequently Asked Questions

What is the equilibrium position of a unit mass hanging from a spring with a spring constant of 1/16 when a sinusoidal force is applied?

The equilibrium position depends on the amplitude and phase of the forcing function; if the force is sinusoidal, the system reaches a steady-state displacement determined by the forcing amplitude and the spring constant.

How does the sinusoidal force 3sin(t) influence the motion of the mass on the spring?

The sinusoidal force causes the mass to oscillate with a driven motion, resulting in a combination of natural oscillation and forced oscillation depending on the frequency and amplitude of the forcing function.

What is the differential equation governing the motion of the mass in this setup?

The equation is m y'' + k y = 3sin(t), where m=1, k=1/16, leading to y'' + (1/16) y = 3sin(t).

What is the natural frequency of the system without forcing?

The natural frequency ω_0 is given by sqrt(k/m) = sqrt(1/16) = 1/4 rad/sec.

How do you find the particular solution for the forced oscillation in this system?

The particular solution can be found using the method of undetermined coefficients, assuming a solution of the form y_p(t) = A sin(t) + B cos(t), and solving for A and B.

What is the amplitude of the steady-state oscillation when the forcing is 3sin(t)?

Since the forcing frequency matches the natural frequency (both are 1 rad/sec), resonance occurs, leading to a potentially large amplitude; the amplitude is proportional to the forcing amplitude divided by the damping factor, which is zero here, indicating unbounded growth in an ideal case.

Does the system exhibit resonance with the given forcing function?

Yes, because the forcing frequency matches the system's natural frequency (both are 1 rad/sec), leading to resonance, which causes the amplitude to increase over time in the absence of damping.

What role does damping play in this system's response to the sinusoidal force?

Damping would limit the amplitude of oscillations at resonance; without damping, the amplitude grows without bound, indicating an idealized, undamped system.

How can the system's response be expressed mathematically?

The total solution is the sum of the homogeneous solution y_h(t) and the particular solution y_p(t), which accounts for the steady-state response to the sinusoidal forcing.

What practical applications relate to understanding this type of forced oscillation system?

This analysis is fundamental in designing mechanical and electrical systems experiencing periodic forces, such as bridges under wind loads, electrical circuits with sinusoidal signals, and vibration analysis in engineering.

Additional Resources

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Investigating the Dynamics of a Mass-Spring System Under Sinusoidal Forcing

Understanding the behavior of a mass hanging from a spring under external forces is fundamental in classical mechanics, with applications spanning engineering, physics, and applied mathematics. In this detailed exploration, we examine a system characterized by a unit mass attached to a spring with a known constant, subjected to a sinusoidal external force. Our goal is to analyze the equilibrium conditions, dynamic responses, and potential resonance phenomena inherent in such a setup.

System Description and Mathematical Modeling

Physical Setup

We consider a mass $m = 1$ unit hanging vertically from an ideal spring with a spring constant $k = \frac{1}{16}$. The mass is initially at the equilibrium position $y=0$, with downward displacement considered positive unless specified otherwise. The system is subjected to a sinusoidal external force:

$$
F(t) = 3 \sin (\omega t),
$$

where ω is the angular frequency of the applied force. The initial condition at $t=0$ is $y(0) = 0$, and the initial velocity $y'(0) = 0$.

Governing Differential Equation

Applying Newton's second law, the sum of forces yields:

$$
m y''(t) + k y(t) = F(t),
$$

which simplifies to:

$$
y''(t) + \frac{k}{m} y(t) = \frac{F(t)}{m}.
$$

Given $m=1$ and $k=\frac{1}{16}$, the differential equation becomes:

$$
\boxed{
y''(t) + \frac{1}{16} y(t) = 3 \sin (\omega t).
}
$$

Analytical Approach to the Differential Equation

Homogeneous Solution

The associated homogeneous equation:

$$
y''(t) + \frac{1}{16} y(t) = 0,
$$

has characteristic roots:

$$
r = \pm i \sqrt{\frac{1}{16}} = \pm i \frac{1}{4}.
$$

Thus, the homogeneous solution is:

$$
y_h(t) = C_1 \cos \left(\frac{t}{4} \right) + C_2 \sin \left(\frac{t}{4} \right),
$$

where C_1, C_2 are determined by initial conditions.

Particular Solution

Since the forcing function is sinusoidal, we seek a particular solution of the form:

$$
y_p(t) = A \sin (\omega t),
$$

or more generally, considering the possibility of resonance, a solution of the form:

$$
y_p(t) = B \sin (\omega t).
$$

Substituting into the differential equation:

$$
-\omega^2 B \sin (\omega t) + \frac{1}{16} B \sin (\omega t) = 3 \sin (\omega t),
$$

which simplifies to:

$$
\left(-\omega^2 + \frac{1}{16} \right) B = 3.
$$

Thus,

$$
B = \frac{3}{\frac{1}{16} - \omega^2} = \frac{3}{\frac{1 - 16 \omega^2}{16}} = \frac{48}{1 - 16 \omega^2}.
$$

Note: When $1 - 16 \omega^2 \neq 0$, this particular solution is valid. If $1 - 16 \omega^2 = 0$, the system is at resonance, requiring a different approach.

General Solution

The complete solution:

$$
y(t) = y_h(t) + y_p(t),
$$

where $y_h(t)$ encapsulates the homogeneous response, and $y_p(t)$ accounts for the particular forced response.

Resonance Analysis and Critical Frequencies

Resonance Condition

Resonance occurs when the denominator in B approaches zero:

$$
1 - 16 \omega^2 = 0 \Rightarrow \omega^2 = \frac{1}{16} \Rightarrow \omega = \pm \frac{1}{4}.
$$

At $\omega = \frac{1}{4}$, the particular solution becomes unbounded, indicating resonance. The physical implication is that the system's oscillations grow without bound if damping is neglected, leading to large amplitude vibrations.

Implications of Forcing Frequency

- For $\omega \neq \frac{1}{4}$: The particular solution is finite, and the system exhibits steady-state oscillations with amplitude:

$$
A(\omega) = \left| \frac{48}{1 - 16 \omega^2} \right|.
$$

$$

- At $\omega = \frac{1}{4}$: Standard particular solutions are invalid, and a modified approach involving multiplication by t is necessary.

Initial Conditions and Complete Solution

Applying Initial Conditions

Given:

$$
y(0) = 0, \quad y'(0) = 0,
$$

we can determine the constants C_1, C_2. The homogeneous solution at $t=0$:

$$
y_h(0) = C_1,
$$

and the derivative:

$$
y_h'(t) = -\frac{1}{4} C_1 \sin \left(\frac{t}{4} \right) + \frac{1}{4} C_2 \cos \left(\frac{t}{4} \right),
$$

which at $t=0$:

$$
y_h'(0) = \frac{1}{4} C_2.
$$

Since the initial displacement and velocity are zero, the particular solution's initial values are considered, and the homogeneous constants are chosen accordingly:

$$
C_1 = - y_p(0) = 0,
$$
$$
C_2 = - 4 y_p'(0),
$$

where $y_p'(t)$ is the derivative of the particular solution.

Full Solution

The complete solution is:

$$
\boxed{
y(t) = C_1 \cos \left(\frac{t}{4} \right) + C_2 \sin \left(\frac{t}{4} \right) + \frac{48}{1 - 16 \omega^2} \sin (\omega t),
}
$$

with constants determined by the initial conditions.

Numerical Exploration and Dynamic Behavior

Sample Numerical Analysis

Suppose the forcing frequency is chosen as $\omega = 0.2$, which is close to the resonance frequency 0.25. The amplitude of the steady-state response:

$$
A(0.2) = \frac{48}{1 - 16 \times 0.2^2} = \frac{48}{1 - 16 \times 0.04} = \frac{48}{1 - 0.64} = \frac{48}{0.36} \approx 133.33,
$$

indicating a significant amplification relative to the forcing amplitude.

Time-Domain Response

Simulating the response over time reveals:

- Transient decay: The homogeneous response diminishes if damping exists; however, damping is not specified, so oscillations persist.
- Steady-state oscillation: The system oscillates with amplitude ≈ 133.33, significantly larger than the initial displacement.

Effect of Damping

In real-world systems, damping terms (e.g., viscous damping with coefficient c) are present:

$$
m y'' + c y' + k y = F(t).
$$

Including damping:

$$
y'' + 2 \zeta \omega_0 y' + \omega_0^2 y = \frac{F(t)}{m},
$$

where:

$$
\omega_0 = \frac{1}{4}, \quad \zeta = \frac{c}{2 m \omega_0}.
$$

Damping reduces the amplitude at resonance, preventing unbounded growth.

Conclusions and Implications

Summary of Findings

- The system is governed by a second-order linear differential equation with sinusoidal forcing.
- The natural frequency of the system is $\frac{1}{4}$.
- The forcing amplitude depends inversely on $1 - 16 \omega^2$, with a divergence at the resonance frequency.
- Near $\omega = \frac{1}{4}$, the system exhibits large amplitude oscillations, emphasizing the importance of damping in real systems.
- The initial conditions influence transient behaviors but do not affect the steady-state response amplitude once transients decay.

Practical Implications

Understanding such systems is crucial in engineering design, where resonance can cause catastrophic failures. It underscores the necessity to:

- Identify natural frequencies.
- Avoid operating at or near resonance frequencies.
- Incorporate damping mechanisms.

Future Directions

Further studies could include:

- Damping effects: Analyzing how damping ratios influence amplitude and transient response.
- Nonlinearities: Considering nonlinear spring forces or large displacements.

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