Alice And Bob Each Choose A Number Uniformly (and Independently) From The Interval [0, 10]. What Is The

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Introduction

In the fascinating world of probability and statistics, simple yet insightful problems often serve as the foundation for understanding more complex concepts. One such classic problem involves two individuals, Alice and Bob, each selecting a number from a continuous interval independently and uniformly at random. Specifically, they both choose a number from the interval [0, 10] without any influence or correlation. This scenario may seem straightforward at first glance, but it opens the door to exploring various probabilistic questions such as the distribution of their differences, sums, and other relationships between their chosen numbers.

Understanding the behavior of such random selections is not only academically interesting but also has practical applications in areas like game theory, decision-making under uncertainty, simulation modeling, and algorithm design. For instance, in auction theory, randomized bidding strategies often assume uniform distributions, while in computer science, randomized algorithms rely on similar principles.

In this article, we will delve deeply into the problem: Alice and Bob each choose a number uniformly and independently from the interval [0, 10]. What is the probability distribution of their difference, sum, and other related quantities? We will analyze these questions step-by-step, employing mathematical rigor, visualizations, and real-world interpretations to enhance understanding.

Understanding the Basic Setup

The Random Variables

Let’s define two random variables:

- A, representing Alice’s chosen number.
- B, representing Bob’s chosen number.

Both A and B are independent and uniformly distributed over the interval $[0, 10]$.

Mathematically, this is expressed as:

$$
A, B \sim \text{Uniform}(0, 10)
$$

$$
\text{with} \quad P(A \in [a, a + da]) = \frac{1}{10} da, \quad P(B \in [b, b + db]) = \frac{1}{10} db
$$

for $a, b \in [0, 10]$.

Joint Probability Density Function (PDF)

Since A and B are independent and uniformly distributed, their joint probability density function (PDF) is:

$$
f_{A,B}(a, b) = f_A(a) \times f_B(b) = \frac{1}{10} \times \frac{1}{10} = \frac{1}{100}
$$

for all $(a, b) \in [0, 10] \times [0, 10]$.

This joint PDF describes a square in the (a, b) plane with side length 10, where the probability density is uniform.

Key Quantities of Interest

Given the random selections of Alice and Bob, several related quantities are often studied:

1. The Difference $D = A - B$

Understanding the distribution of the difference between their chosen numbers is fundamental in probability problems. It informs us about the likelihood that Alice's number exceeds Bob's, or vice versa, and the typical magnitude of their difference.

2. The Sum $S = A + B$

The distribution of the sum helps in various applications, such as probability of total outcomes, ranges, and bounds.

3. The Absolute Difference $|A - B|$

Often, the absolute difference is of interest in measuring how close or far apart their choices are.

4. The Range of Possible Values

Since both A and B are in $[0, 10]$, the sum and difference have specific bounds:

- Sum $S \in [0, 20]$
- Difference $D \in [-10, 10]$
- Absolute difference $|A - B| \in [0, 10]$

Distribution of the Difference $D = A - B$

Geometric Interpretation

The joint PDF is uniform over a square with corners at $(0,0)$, $(0,10)$, $(10,0)$, and $(10,10)$.

The difference $D = A - B$ takes values in $[-10, 10]$. To find the distribution of D, we analyze the region in the (a, b) plane where $a - b = d$.

Deriving the PDF of D

For a fixed value $d \in [-10, 10]$, the set of points where $a - b = d$ forms a line:

$$
b = a - d
$$

Within the square $[0, 10] \times [0, 10]$, the intersection of this line with the square determines the probability density.

The probability density function $f_D(d)$ is proportional to the length of the intersection between the line $a - b = d$ and the square.

Step-by-Step Calculation

1. For each fixed $d \in [-10, 10]$, the corresponding line segment within the square has a length:

$$
L(d) = \text{length of the intersection of } b = a - d \quad \text{with} \quad 0 \leq a, b \leq 10
$$

2. Find the endpoints of this segment:

- When $a = 0$, $b = -d$. Since $b \geq 0$, this requires $-d \geq 0 \Rightarrow d \leq 0$.

- When $a = 10$, $b = 10 - d$. Since $b \leq 10$, this requires $10 - d \leq 10$, or $d \geq 0$.

Similarly, for $b = 0$, $a = d$, and for $b = 10$, $a = 10 + d$.

3. Based on these constraints, the length $L(d)$ is:

$$
L(d) =
\begin{cases}
10 + d, & \text{if } -10 \leq d \leq 0 \\
10 - d, & \text{if } 0 \leq d \leq 10
\end{cases}
$$

4. Since the joint PDF is uniform over the square, the probability density function of D is proportional to $L(d)$:

$$
f_D(d) = \frac{L(d)}{\text{area of the square}} = \frac{L(d)}{100}
$$

Therefore,

$$
f_D(d) =
\begin{cases}
\frac{10 + d}{100}, & -10 \leq d \leq 0 \\
\frac{10 - d}{100}, & 0 \leq d \leq 10
\end{cases}
$$

Final Expression for $f_D(d)$

$$
f_D(d) = \frac{1}{100} \times
\begin{cases}
10 + d, & -10 \leq d \leq 0 \\
10 - d, & 0 \leq d \leq 10 \\
0, & \text{otherwise}
\end{cases}
$$

This is a symmetric "tent-shaped" distribution centered at zero, with maximum at $d=0$.

Visualization

Plotting $f_D(d)$ reveals a triangle with peak at $d=0$:

- At $d=0$, $f_D(0) = \frac{10}{100} = 0.1$.

- At the extremes $d = \pm 10$, $f_D(d) = 0$.

Expected Value and Variance

- Mean of D:

Since the distribution is symmetric around zero,

$$
E[D] = 0
$$

- Variance of D:

Calculating explicitly,

$$
Var(D) = E[D^2] - (E[D])^2 = E[D^2]
$$

Compute $E[D^2]$:

$$
E[D^2] = \int_{-10}^{10} d^2 f_D(d) dd
$$

Using the piecewise form:

$$
E[D^2] = \int_{-10}^{0} d^2 \frac{10 + d}{100} dd + \int_{0}^{10} d^2 \frac{10 - d}{100} dd
$$

Calculating these integrals yields the variance.

Distribution of the Sum $S = A + B$

Geometric Approach

The sum S ranges from 0 to 20. For a fixed $s \in [0, 20]$, the set of points satisfying $a + b = s$ is a line segment in the (a, b) plane.

Deriving the PDF of S

1. For $s \in [0, 10]$:

- The line segment ranges from $(a, b) = (0, s)$ to $(a, s, 0)$, with a from 0 to s.

- The length of the segment:

\[
L(s) = \sqrt{(a_{end} - a_{start})^2 + (b

Frequently Asked Questions

What is the expected value of Alice's chosen number?

The expected value of Alice's number is 5, since she chooses uniformly from [0, 10].

What is the expected value of the difference between Alice's and Bob's numbers?

The expected absolute difference between the two numbers is approximately 3.33.

What is the probability that Alice's number is greater than Bob's?

The probability that Alice's number exceeds Bob's is 0.5, due to symmetry.

What is the distribution of the difference between Alice's and Bob's numbers?

The difference follows a triangular distribution over [-10, 10], with a peak at 0.

What is the probability that Alice's number and Bob's number are within 2 units of each other?

The probability that their numbers differ by less than 2 is approximately 0.36.

How do the distributions of Alice's and Bob's numbers compare?

Both are uniformly distributed over [0, 10], so their distributions are identical and constant across the interval.

What is the probability that both Alice and Bob

choose numbers above 7?

The probability that both pick numbers above 7 is 0.09.

What is the variance of the difference between Alice's and Bob's numbers?

The variance of the difference is approximately 16.67.

If Alice's number is at least 8, what is the expected value of Bob's number?

Given Alice's number is at least 8, the expected value of Bob's number remains 5, since the choices are independent and uniform.

Additional Resources

Alice And Bob Each Choose A Number Uniformly (and Independently) From The Interval [0, 10]. What Is The

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Introduction

In the realm of probability theory and statistical analysis, seemingly simple scenarios often conceal rich, complex insights. One such scenario involves two individuals—Alice and Bob—each independently selecting a number from a continuous uniform distribution over the interval [0, 10]. While the setting appears straightforward, the question of what statistical properties emerge from such a process, especially concerning the difference between their chosen numbers, opens the door to fascinating explorations in expectation, variance, distributional characteristics, and their implications for broader probabilistic models.

This article aims to thoroughly investigate the problem: "Alice and Bob each choose a number uniformly (and independently) from the interval [0, 10]. What is the distribution and expected value of the difference between their choices?" We will dissect this question step-by-step, analyze the mathematical underpinnings, and explore potential extensions and applications relevant to fields such as game theory, decision analysis, and statistical modeling.

The Fundamental Setup

Defining the Random Variables

Let us denote Alice's choice as X and Bob's choice as Y. Both X and Y are independent random variables, each uniformly distributed over [0, 10]:

$$
X \sim \text{Uniform}(0, 10), \quad Y \sim \text{Uniform}(0, 10)
$$

The probability density functions (pdf) for both are:

$$
f_X(x) = \frac{1}{10}, \quad 0 \leq x \leq 10
$$
$$
f_Y(y) = \frac{1}{10}, \quad 0 \leq y \leq 10
$$

Because of independence, the joint pdf is simply:

$$
f_{X,Y}(x, y) = f_X(x) \times f_Y(y) = \frac{1}{100}
$$
for $x, y \in [0, 10]$.

Key Question: Distribution of the Difference $D = |X - Y|$

The core of the investigation centers on understanding the distribution of the absolute difference:

$$
D = |X - Y|
$$

This quantity is of interest in various contexts, such as measuring similarity, divergence, or in decision-making scenarios.

Why Focus on the Absolute Difference?

The absolute difference captures the magnitude of the discrepancy between Alice's and Bob's choices, regardless of who chose the larger number. This is particularly relevant in social choice, consensus modeling, or conflict resolution contexts, where the magnitude of disagreement matters more than who is higher or lower.

Distribution of the Difference $X - Y$

Before analyzing $D = |X - Y|$, it is instructive to examine the distribution of the difference without the absolute value:

$$
Z = X - Y
$$

Understanding Z will facilitate deriving the distribution of D.

Deriving the Distribution of $Z = X - Y$

Since X and Y are independent and uniform over [0, 10], the joint pdf $f_{X,Y}(x, y)$ is uniform over the square $[0,10] \times [0,10]$.

The variable $Z = X - Y$ takes values in $[-10, 10]$, because:

- The maximum difference occurs when $X=10, Y=0$, giving $Z=10$.
- The minimum difference occurs when $X=0, Y=10$, giving $Z=-10$.

Distribution of Z

The pdf of Z, $f_Z(z)$, is obtained through convolution:

$$
f_Z(z) = \int_{-\infty}^\infty f_X(x) f_Y(x - z) dx
$$

However, since f_X and f_Y are uniform over [0, 10], the convolution reduces to integrating over the intersection of the domains.

Alternatively, because the joint distribution is uniform over a square, the distribution of the difference corresponds to the distribution of the difference of two independent uniform variables, which is well-studied.

Distribution of the Difference of Two Independent Uniforms

The probability density function of the difference $Z = X - Y$, where $X, Y \sim \text{Uniform}(a, b)$, is known:

$$
f_Z(z) = \frac{1}{(b - a)^2} \times \text{length of the intersection of the regions satisfying } x - y = z
$$

In our case, $a=0$, $b=10$.

The distribution of Z over $[-10, 10]$ is a triangular distribution:

$$
f_Z(z) = \frac{1}{100} (10 - |z|), \quad -10 \leq z \leq 10
$$

Derivation Sketch:

1. For a fixed z, the set of (x, y) such that $x - y = z$ lies along the line $y = x - z$.
2. The intersection of this line with the square $[0, 10] \times [0, 10]$ forms a segment.
3. The length of this segment varies with z, leading to the triangular shape.

The resulting pdf:

$$
f_Z(z) = \frac{1}{100} (10 - |z|), \quad -10 \leq z \leq 10
$$

This is a symmetric triangular distribution with the peak at $z=0$.

Distribution of the Absolute Difference $D = |Z|$

Since $D = |Z|$, its distribution is derived from $f_Z(z)$:

$$
f_D(d) = f_Z(d) + f_Z(-d)
$$

for $d \in [0, 10]$.

Given the symmetry of $f_Z(z)$:

$$
f_D(d) = 2 \times f_Z(d) = 2 \times \frac{1}{100} (10 - d) = \frac{2}{100} (10 - d) = \frac{1}{50}(10 - d)
$$

for $0 \leq d \leq 10$.

Result:

$$
\boxed{
f_D(d) = \frac{1}{50}(10 - d), \quad 0 \leq d \leq 10
}
$$

This is a triangular distribution over $[0, 10]$, with the highest density at zero and decreasing linearly to zero at $d=10$.

Expected Value of the Absolute Difference D

The expected value $E[D]$ can be computed directly:

$$
E[D] = \int_0^{10} d \times f_D(d) dd = \int_0^{10} d \times \frac{1}{50} (10 - d) dd
$$

Simplify the integral:

$$
E[D] = \frac{1}{50} \int_0^{10} d (10 - d) dd
$$

Compute the integral:

$$
\int_0^{10} d(10 - d) dd = \int_0^{10} (10d - d^2) dd
$$

$$
= 10 \int_0^{10} d dd - \int_0^{10} d^2 dd
$$

$$
= 10 \left[\frac{d^2}{2}\right]_0^{10} - \left[\frac{d^3}{3}\right]_0^{10}
$$

$$
= 10 \times \frac{100}{2} - \frac{1000}{3} = 10 \times 50 - \frac{1000}{3} = 500 - \frac{1000}{3}
$$

Expressed as a single fraction:

$$
= \frac{1500 - 1000}{3} = \frac{500}{3}
$$

Now, multiply by $\frac{1}{50}$:

$$
E[D] = \frac{1}{50} \times \frac{500}{3} = \frac{500}{150} = \frac{10}{3} \approx 3.333
$$

```
Result:

\[
\boxed{
E[D] = \frac{10}{3} \approx 3.333
}
\]

This indicates that on average, the absolute difference between Alice’s and
Bob’s choices is roughly 3.33 units.

---

Variance of the Absolute Difference \(D\)

To compute the variance \(Var(D)\), we need \(E[D^2]\):

\[
E[D^2] = \int_0^{10} d^2 \times f_D(d) dd = \frac{1}{50} \int_0^{10} d^2 (10
- d) dd
\]

Compute the integral:

\[
\int_0^{10} d^2 (10 - d) dd = 10 \int_0^{10} d^2 dd - \int_0^{10} d^3 dd
\]

Calculate each:

\[
10 \times \left[\
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