All Polynomials Of Degree At Most 3 With Integer Coefficients. Determine If The Given Set Is A Subspace

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Understanding the structure of polynomial spaces is a fundamental aspect of linear algebra and abstract algebra. Specifically, when dealing with polynomials of degree at most 3 with integer coefficients, it’s essential to analyze their properties, particularly whether certain subsets form subspaces within the larger polynomial space. This article explores the set of all such polynomials, investigates the criteria for subspace formation, and provides comprehensive insights into the mathematical principles involved.

Introduction to Polynomial Spaces

Polynomials are algebraic expressions consisting of variables and coefficients combined using addition, subtraction, and multiplication. The set of all polynomials with coefficients in a particular field (such as real numbers ℝ or integers ℤ) forms a vector space under polynomial addition and scalar multiplication.

Polynomial Space of Degree At Most 3

The space of all polynomials of degree at most 3 with real coefficients is typically denoted as $P_3(\mathbb{R})$. Its elements take the form:

$$ p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 $$

where $a_0, a_1, a_2, a_3 \in \mathbb{R}$.

However, our focus is on polynomials with integer coefficients:

$$ p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3, \quad \text{with } a_i \in \mathbb{Z} $$

This set is often denoted as $P_3(\mathbb{Z})$.

Defining the Set of Interest

The set under consideration is:

$$ V = \{ p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_i \in \mathbb{Z} \} $$

This set includes all polynomials of degree at most 3 with integer coefficients.

Key Concepts: Subspace Criteria

Before determining whether V is a subspace, let's recall the fundamental criteria:

Subspace of a Vector Space

A subset W of a vector space V over a field F is a subspace if and only if:

1. Zero Vector Inclusion: The zero vector of V is in W.
2. Closed Under Addition: For all $u, v \in W$, $u + v \in W$.
3. Closed Under Scalar Multiplication: For all $u \in W$ and $c \in F$, $c u \in W$.

In our context, these translate to:

- The zero polynomial (all coefficients zero) with integer coefficients must be in V.
- The sum of any two polynomials in V must also be in V.
- Multiplying any polynomial in V by any scalar (from the field over which the space is defined) must result in a polynomial still in V.

Analyzing the Set of Polynomials with Integer Coefficients

Let's explore whether the set V satisfies these criteria.

1. Zero Vector Inclusion

The zero polynomial:

$$ p_0(x) = 0 + 0x + 0x^2 + 0x^3 $$

has all coefficients zero, which are integers. Therefore, $p_0(x) \in V$. The first criterion is satisfied.

2. Closure Under Addition

Suppose $p_1(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ and $p_2(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$, with all $a_i, b_i \in \mathbb{Z}$.

Their sum:

$$p_1(x) + p_2(x) = (a_0 + b_0) + (a_1 + b_1) x + (a_2 + b_2) x^2 + (a_3 + b_3) x^3$$

Since integers are closed under addition, all coefficients remain integers. Thus, the sum is in V.

3. Closure Under Scalar Multiplication

This is where the critical issue arises.

- The set V consists of polynomials with integer coefficients.
- Scalar multiplication involves multiplying a polynomial by a scalar c.

Key Point: When considering V as a subset of a vector space over $\mathbb{R}$ or $\mathbb{Q}$, scalar multiplication by any scalar in these fields should be closed within V.

Analysis:

- If scalar $c \in \mathbb{R}$, then $c p(x)$ will generally have coefficients $c a_i$.
- For $c p(x)$ to be in V, all coefficients $c a_i$ must be integers.

Implication:

- Since $a_i \in \mathbb{Z}$, $c a_i \in \mathbb{Z}$ only if $c \in \mathbb{Z}$ (or if c divides all a_i in some special cases).

Conclusion:

- Scalar multiplication by arbitrary real scalars does not necessarily result in a polynomial with integer coefficients.
- Therefore, V is not closed under scalar multiplication by arbitrary real scalars.

Implications for Subspace Structure

Given the above analysis, the set V of all polynomials of degree at most 3 with integer coefficients is not a subspace of the polynomial vector space over $\mathbb{R}$ or $\mathbb{Q}$.

However, if we restrict the scalar field to the integers $\mathbb{Z}$, then scalar multiplication within V remains closed:

- For $c \in \mathbb{Z}$, $c p(x)$ also has integer coefficients.
- In this context, V would be a submodule over $\mathbb{Z}$, but not a vector space over $\mathbb{R}$.

Summary: Is the Set a Subspace?

Criterion	Satisfied?	Explanation
Zero vector	Yes	The zero polynomial is in V.
Closed under addition	Yes	Sum of two integer-coefficient polynomials remains in V.
Closed under scalar multiplication over $\mathbb{R}$	No	Multiplying by arbitrary real scalars can produce non-integer coefficients.
Closed under scalar multiplication over $\mathbb{Z}$	Yes	Multiplying by integers preserves integer coefficients.

Final Verdict:

- As a subset of polynomials over $\mathbb{R}$, V is not a subspace because it fails closure under scalar multiplication by arbitrary real numbers.
- Over the integers $\mathbb{Z}$, V forms a submodule, but not a vector space.

Applications and Significance in Mathematics

Understanding the structure of polynomial spaces with integer coefficients is crucial in various mathematical domains:

- Number Theory: Studying integer polynomials relates to Diophantine equations and algebraic integers.
- Algebraic Geometry: Polynomial rings over integers are fundamental in defining algebraic varieties over $\mathbb{Z}$.
- Computational Algebra: Integer coefficient polynomials are essential in algorithms requiring exact arithmetic, such as in computer algebra systems.

Conclusion

Analyzing the set of all polynomials of degree at most 3 with integer coefficients reveals important insights into the nature of algebraic structures. While this set includes the zero polynomial and is closed under addition, it fails to be a subspace over the real or rational fields due to scalar multiplication issues. Recognizing these distinctions is vital for mathematicians working in abstract algebra, linear algebra, and related fields.

Understanding the criteria for subspace formation helps clarify the algebraic properties of polynomial sets and guides the appropriate application of these concepts in theoretical and applied mathematics.

Further Reading

- Linear Algebra by Gilbert Strang
- Abstract Algebra by David S. Dummit and Richard M. Foote
- Introduction to Polynomial Algebra in algebraic structures and ring theory

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Keywords: polynomials, integer coefficients, polynomial space, subspace criteria, linear algebra, algebraic structures, polynomial rings, scalar multiplication, submodule, vector space, degree at most 3

Frequently Asked Questions

What are the criteria to determine if the set of all polynomials of degree at most 3 with integer coefficients forms a subspace?

The set must be closed under addition and scalar multiplication, contain the zero polynomial, and satisfy the properties of a vector space. Since the set consists of polynomials with integer coefficients, closure under scalar multiplication is only guaranteed when scalars are integers, so the set forms a subspace over the integers but not over the reals.

Does the set of all degree at most 3 polynomials with integer coefficients include the zero polynomial? Why is this important?

Yes, the zero polynomial (all coefficients zero) has integer coefficients and degree at most 3. Including the zero polynomial is essential because it is a requirement for any subset to be a subspace, serving as the additive identity.

Is the set of all polynomials of degree at most 3 with integer coefficients closed under addition? Provide an explanation.

Yes, the sum of two polynomials with integer coefficients and degree at most 3 also has integer coefficients and degree at most 3, so the set is closed under addition.

Can scalar multiplication by arbitrary real numbers preserve the set of polynomials with integer coefficients? Why or why not?

No, scalar multiplication by arbitrary real numbers can produce polynomials with non-integer coefficients, so the set is not closed under scalar multiplication over the real numbers. It is only closed under multiplication by integers.

Given the set of all polynomials of degree at most 3 with integer coefficients, over which field does this set form a subspace?

The set forms a subspace over the field of integers if considering scalar multiplication only by integers. Over the real numbers or rationals, it is not a subspace because it is not closed under scalar multiplication.

How can we modify the set to ensure it forms a subspace over the real numbers?

To ensure the set forms a subspace over the real numbers, we need to allow polynomials with real coefficients, not just integers. If the coefficients are restricted to integers, the set is only a subspace over the integers, not the reals.

Additional Resources

All Polynomials Of Degree At Most 3 With Integer Coefficients: Determine If The Given Set Is A Subspace

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Introduction

In the realm of linear algebra and vector spaces, understanding the structure of various sets of mathematical objects is fundamental. One such set is the collection of all polynomials of degree at most 3 with integer coefficients. This set is often denoted as:

$$V = \{ p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_i \in \mathbb{Z} \}$$

where a_0, a_1, a_2, a_3 are integers.

The central question addressed in this review is: Is this set a subspace of the vector space of all polynomials with real coefficients? To answer this, we need to explore the properties that define a subspace, analyze the particular features of the set V, and determine whether it satisfies these properties.

Understanding the Underlying Vector Space

Before analyzing the specific set, it's crucial to understand the ambient space within which we are examining V.

The Space of All Polynomials with Real Coefficients

- The set of all polynomials with real coefficients, denoted $P(\mathbb{R})$, is a vector space over $\mathbb{R}$.
- Elements are of the form $p(x) = \sum_{i=0}^{n} c_i x^i$, with $c_i \in \mathbb{R}$.
- Operations:
- Addition: $p(x) + q(x) = \sum_{i=0}^{n} (c_i + d_i) x^i$
- Scalar multiplication: For $\lambda \in \mathbb{R}$, $\lambda p(x) = \sum_{i=0}^{n} \lambda c_i x^i$.

This space is well-understood, infinite-dimensional, and contains many interesting subspaces.

Defining the Set V: Polynomials of Degree At Most 3 with Integer Coefficients

The set in question can be explicitly written as:

$$V = \left\{ p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_i \in \mathbb{Z} \right\}$$

This set has the following features:

- Finite-dimensionality: As polynomials are degree at most 3, the set is spanned by the basis:

$$\{ 1, x, x^2, x^3 \}$$

- Integer coefficients: The coefficients are restricted to integers, which is a key point influencing the subspace property.

Properties of a Subspace: Criteria and Analysis

To determine if V is a subspace of $P(\mathbb{R})$, it must satisfy three fundamental criteria:

1. Contains the zero vector: The zero polynomial $p(x) = 0$ must be in V.
2. Closed under addition: For any $p(x), q(x) \in V$, their sum $p(x) + q(x)$ must also be in V.
3. Closed under scalar multiplication: For any $p(x) \in V$ and $\lambda \in \mathbb{R}$, the product $\lambda p(x)$ must also be in V.

Let's analyze each criterion carefully in the context of V.

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1. Zero Polynomial in V

- The zero polynomial 0 has all coefficients zero: $0 + 0x + 0x^2 + 0x^3$.
- Since zero is an integer, $0 \in \mathbb{Z}$, so the zero polynomial is in V.

Conclusion: The first condition is satisfied.

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2. Closure Under Addition

- Take any two polynomials $p(x), q(x) \in V$:

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$
$$q(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$$

where $a_i, b_i \in \mathbb{Z}$.

- Their sum:

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1) x + (a_2 + b_2) x^2 + (a_3 + b_3) x^3$$

- Since integers are closed under addition, each coefficient $a_i + b_i \in \mathbb{Z}$.

Conclusion: Closure under addition holds within V.

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3. Closure Under Scalar Multiplication

This is where the critical issue arises. Consider scalar multiplication:

- For $p(x) \in V$ and scalar $\lambda \in \mathbb{R}$:

$$\lambda p(x) = \lambda a_0 + \lambda a_1 x + \lambda a_2 x^2 + \lambda a_3 x^3$$

- For $\lambda p(x)$ to be in V, all coefficients must be integers:

$$\lambda a_i \in \mathbb{Z} \quad \text{for all } i=0,1,2,3$$

- Since $a_i \in \mathbb{Z}$, this requires:

$$\lambda a_i \in \mathbb{Z} \quad \Rightarrow \quad \lambda \in \frac{\mathbb{Z}}{a_i}$$

- But a_i can be any integer, and λ can be any real number. Therefore, unless λ is an integer itself, this condition generally fails.

Counterexample:

- Take $p(x) = 1 + 0x + 0x^2 + 0x^3$, with coefficients in $\mathbb{Z}$.
- Multiply by $\lambda = \frac{1}{2}$:

$$\frac{1}{2} p(x) = \frac{1}{2} + 0x + 0x^2 + 0x^3$$

- The coefficient $\frac{1}{2}$ is not an integer, so $\frac{1}{2} p(x) \notin V$.

Conclusion: Closure under scalar multiplication over $\mathbb{R}$ does not hold for V.

Implications of the Scalar Multiplication Issue

The failure of closure under scalar multiplication by arbitrary real scalars indicates that:

- V is not a subspace of $P(\mathbb{R})$ when considering the vector space over $\mathbb{R}$.

This is a crucial point because the definition of a subspace requires the scalar field to be the same as the one over which the larger space is defined—in this case, the real numbers.

Alternative Perspectives and Subspaces Within \(V \)

Although \(V \) is not a subspace of \(P(\mathbb{R}) \), it can be viewed as a subspace over the field of rational numbers \(\mathbb{Q} \):

- If the scalar field is \(\mathbb{Q} \), then scalar multiplication of a polynomial with integer coefficients by a rational number yields a polynomial with rational coefficients, which can be outside \(V \) unless the scalar is an integer.

- However, the set \(V \) considered as a subset of polynomials with rational coefficients and with coefficients in \(\mathbb{Z} \) forms a submodule over \(\mathbb{Z} \), but not a subspace over \(\mathbb{Q} \).

- Key point: To ensure \(V \) is a subspace, the scalar field must be compatible with the set's coefficient domain, which is \(\mathbb{Z} \).

- Over \(\mathbb{Z} \), the set forms a module, not a vector space.

Summary of Findings

- The set of all polynomials of degree at most 3 with integer coefficients is closed under addition.
- It contains the zero polynomial.
- It fails closure under scalar multiplication over \(\mathbb{R} \) because multiplying by arbitrary real scalars produces coefficients outside \(\mathbb{Z} \).

Therefore:

The set \(V \) is not a subspace of the vector space of all polynomials with real coefficients.

However:

- If the scalar field is taken as \(\mathbb{Z} \) (which is not a field), \(V \) forms a module—a structure similar to a vector space but over a ring instead

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