An Urn Contains Four Balls Numbered 1, 2, 3, And 4. If Two Balls Are Drawn From The Urn At Random (that

An Urn Contains Four Balls Numbered 1, 2, 3, And 4. If Two Balls Are Drawn From The Urn At Random (that is a classic problem in probability theory that helps illustrate fundamental concepts such as sample space, probability calculations, and combinatorial analysis. This scenario is often used to introduce students and enthusiasts to the principles of randomness, equally likely outcomes, and how to compute probabilities in simple experiments involving drawing objects without replacement. Understanding this problem lays the groundwork for more complex probabilistic models and real-world applications, from quality control to game theory.

In this comprehensive article, we will explore the problem in detail, analyze the sample space, calculate various probabilities, and discuss extensions and related concepts. Whether you're a student studying probability, a teacher preparing lessons, or someone curious about how probabilities are computed in everyday situations, this article aims to clarify the concepts with clear explanations and structured analysis.

Understanding the Problem Setup

The Basic Scenario

Imagine an urn that contains four distinct balls, each labeled with a unique number: 1, 2, 3, and 4. The balls are identical in size and shape but are distinguishable by their labels. The process involves drawing two balls from the urn at random, without replacement, meaning once a ball is drawn, it is not put back before drawing the second ball.

The key questions that typically arise in this scenario include:

- What is the total number of possible outcomes when two balls are drawn?
- What are the probabilities of specific outcomes, such as drawing a particular ball or obtaining certain combinations?
- How do these probabilities change if the drawing process is with replacement?

The Importance of the Assumptions

The problem assumes:

- Equal likelihood of each ball being drawn in any draw (fairness).
- No replacement after the first draw, meaning the second draw is from the remaining three balls.
- Random selection, with each pair of balls equally likely.

These assumptions are vital because they determine how we compute the sample space and probabilities. Altering any of these assumptions can lead to different probability models.

Sample Space and Possible Outcomes

Enumerating Outcomes without Replacement

Since we are drawing two balls from four without replacement, the total number of possible outcomes can be calculated using combinatorics.

- Number of possible pairs: The number of ways to choose 2 balls from 4 is given by the combination formula:

$$
\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4 \times 3}{2 \times 1} = 6
$$

- Explicit list of outcomes:

1. (1, 2)
2. (1, 3)
3. (1, 4)
4. (2, 3)
5. (2, 4)
6. (3, 4)

Each of these pairs can be considered an equally likely outcome if the drawing process is random and fair.

Order Matters or Not?

In many cases, the order in which the balls are drawn may or may not matter:

- Order matters (permutations): If the sequence of drawing is important, then the total number of outcomes is the number of permutations of 4 items taken 2 at a time:

$$
P(4, 2) = 4 \times 3 = 12
$$

- Order does not matter (combinations): If only the pair of balls drawn is important, regardless of the order, then the total outcomes are the 6 combinations listed earlier.

In probability calculations, the context often clarifies whether order matters. For this problem, since the question is about "drawing two balls" in a single event, it's typical to consider combinations where order is not significant unless specified.

Calculating Probabilities in the Draws

Probability of Drawing a Specific Ball

Suppose you are interested in the probability that a particular ball, say ball 2, is drawn in the two draws.

- Event: Ball 2 is among the two balls drawn.
- Number of favorable outcomes:

For Ball 2 to be included, the other ball can be any of the remaining three (1, 3, 4). The pairs containing ball 2 are:

1. (2, 1)
2. (2, 3)
3. (2, 4)

- Total outcomes: 6 (from the combination list).

- Probability:

$$
P(\text{ball 2 is drawn}) = \frac{\text{favorable outcomes}}{\text{total outcomes}} = \frac{3}{6} = \frac{1}{2}
$$

Thus, there is a 50% chance that ball 2 will be among the two balls drawn.

Probability of Drawing a Specific Pair

If you want to find the probability of drawing a specific pair, say balls 1 and 3:

- Number of favorable outcomes: 1 (the pair (1, 3))
- Total outcomes: 6
- Probability:

$$
P(\text{drawing balls 1 and 3}) = \frac{1}{6}
$$

This illustrates that each pair has an equal probability in the case of a uniform random draw without replacement.

Probability of Specific Events

Other interesting probabilities include:

- Drawing at least one particular ball (e.g., ball 4).
- Drawing two balls both greater than 2 (i.e., 3 and 4).
- The probability of drawing two even-numbered balls (2 and 4).

Let's analyze some of these:

1. Probability of drawing at least one ball numbered 4:

- Favorable outcomes:
- Pairs involving ball 4: (1, 4), (2, 4), (3, 4)
- Total: 3 outcomes
- Probability:

$$
P(\text{at least one 4}) = \frac{3}{6} = \frac{1}{2}
$$

2. Probability of drawing both balls greater than 2:

- The balls greater than 2 are 3 and 4.
- Favorable outcome:
- Only one pair: (3, 4)
- Probability:

$$
P(\text{both > 2}) = \frac{1}{6}
$$

3. Probability of drawing two even-numbered balls:

- Even balls are 2 and 4.
- Favorable pair:
- (2, 4)
- Probability:

$$
P(\text{both even}) = \frac{1}{6}
$$

These calculations demonstrate how to analyze different events within the same probability space.

Extension: Drawing With Replacement

The previous analysis assumes no replacement; once a ball is drawn, it is not put back into the urn. If the process involves drawing with replacement—meaning after drawing the first ball, it's returned to the urn before the second draw—the probabilities change.

Sample Space with Replacement

- Total possible ordered outcomes: 16, because each of the two draws can be any of the four balls independently:

$$
4 \times 4 = 16
$$

- Each outcome can be represented as an ordered pair, for example, (1, 3).

Probability Calculations with Replacement

- The probability of drawing a particular ball in a single draw remains $\frac{1}{4}$.
- For example, the probability of drawing ball 2 twice in a row:

$$
P(\text{ball 2 on first and second draw}) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}
$$

- The probability of drawing at least one ball numbered 2 in two draws:

- Complement approach:

$$
P(\text{no 2 in both draws}) = \left(\frac{3}{4} \right)^2 = \frac{9}{16}
$$

$$
P(\text{at least one 2}) = 1 - \frac{9}{16} = \frac{7}{16}
$$

This illustrates how the probability model adjusts when the sampling process involves replacement.

Applications of the Urn Drawing Problem

The simple urn drawing problem is a microcosm of many real-world situations. Here are some fields and scenarios where similar models are applied:

Quality Control in Manufacturing

- Randomly selecting items from a batch to check for defects.
- Calculating the probability of detecting faulty items based on sampling.

Card Games and Gambling

- Drawing cards from a deck and calculating odds of specific hands.
- Understanding probabilities in games like poker or blackjack.

Statistics and Sampling Techniques

- Designing experiments and surveys with random sampling.
- Estimating population parameters based on sample outcomes.