Circle The Implicit Solution To The Differential Equation Y' + 2y = 7x Y(0) = 0 A. 4y + 7 = 14x + 7e-2x

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Understanding differential equations is fundamental in mathematics, physics, engineering, and many applied sciences. These equations describe the relationship between a function and its derivatives, encapsulating how systems evolve over time or space. Among the various methods to solve differential equations, implicit solutions often provide a more comprehensive view of the relationship between variables, especially when explicit solutions are difficult to obtain. This article explores the process of finding an implicit solution to the differential equation:

$$ Y' + 2y = 7x, \quad \text{with initial condition} \quad Y(0) = 0, $$

and examines how this solution relates to the given implicit relation:

$$ 4y + 7 = 14x + 7e^{-2x}. $$

Through a detailed analysis, step-by-step solution procedures, and contextual insights, we aim to illuminate the significance and application of implicit solutions in solving first-order linear differential equations.

Understanding the Differential Equation $Y' + 2y = 7x$

Form of the Differential Equation

The given differential equation:

$$ Y' + 2y = 7x $$

is a first-order linear differential equation. It can be written in the standard form:

$$ \frac{dy}{dx} + p(x) y = q(x), $$

where:

- $p(x) = 2$,
- $q(x) = 7x$.

This form facilitates the use of an integrating factor to find solutions.

Initial Condition and Its Significance

The initial condition:

$$Y(0) = 0$$

sets a specific point through which the solution must pass. It is crucial for determining the constant of integration when solving the differential equation explicitly.

Solving the Differential Equation Using Integrating Factor

Step 1: Find the Integrating Factor

The integrating factor $\mu(x)$ for a first-order linear differential equation is:

$$\mu(x) = e^{\int p(x) dx} = e^{\int 2 dx} = e^{2x}.$$

Step 2: Multiply the Entire Equation by the Integrating Factor

Multiplying both sides:

$$e^{2x} \frac{dy}{dx} + 2 e^{2x} y = 7x e^{2x}.$$

Note that the left side simplifies to the derivative of $y e^{2x}$:

$$\frac{d}{dx} \left(y e^{2x} \right) = 7x e^{2x}.$$

Step 3: Integrate Both Sides

Integrating:

$$ y e^{2x} = \int 7x e^{2x} dx + C. $$

The integral on the right involves integration by parts.

Step 4: Compute the Integral $\int 7x e^{2x} dx$

Let's compute:

$$ I = \int x e^{2x} dx. $$

Using integration by parts:

- $u = x \quad \Rightarrow \quad du = dx$,
- $dv = e^{2x} dx \quad \Rightarrow \quad v = \frac{1}{2} e^{2x}$.

Then:

$$ I = x \cdot \frac{1}{2} e^{2x} - \int \frac{1}{2} e^{2x} dx = \frac{x}{2} e^{2x} - \frac{1}{2} \cdot \frac{1}{2} e^{2x} + C = \frac{x}{2} e^{2x} - \frac{1}{4} e^{2x} + C. $$

Multiplying by 7:

$$ \int 7x e^{2x} dx = 7 I = 7 \left(\frac{x}{2} e^{2x} - \frac{1}{4} e^{2x} \right) + C = \frac{7x}{2} e^{2x} - \frac{7}{4} e^{2x} + C. $$

Step 5: Write the General Solution

Recall:

$$ y e^{2x} = \frac{7x}{2} e^{2x} - \frac{7}{4} e^{2x} + C. $$

Divide both sides by e^{2x}:

$$ y = \frac{7x}{2} - \frac{7}{4} + C e^{-2x}. $$

Applying the initial condition $y(0) = 0$:

$$ 0 = \frac{7 \cdot 0}{2} - \frac{7}{4} + C e^{0} \Rightarrow 0 = 0 - \frac{7}{4} + C \Rightarrow C = \frac{7}{4}. $$

Thus, the particular solution is:

$$ y(x) = \frac{7x}{2} - \frac{7}{4} + \frac{7}{4} e^{-2x}. $$

Expressing the Solution in Implicit Form

While the explicit form provides a direct formula for y, sometimes it's more insightful to express the solution implicitly, especially when dealing with complex relationships or when the explicit form is unwieldy.

Given the explicit solution:

$$ y(x) = \frac{7x}{2} - \frac{7}{4} + \frac{7}{4} e^{-2x}, $$

we can manipulate this to relate y and x without explicitly solving for y.

Multiply both sides by 4:

$$ 4 y = 14 x - 7 + 7 e^{-2x}. $$

Rearranged:

$$ 4 y + 7 = 14 x + 7 e^{-2x}. $$

This gives the implicit relation as:

$$ 4 y + 7 = 14 x + 7 e^{-2x}. $$

This form matches the implicitly given relation in the problem statement, providing a direct link between the solution and the implicit equation.

Significance and Applications of Implicit Solutions

Why Use Implicit Solutions?

Implicit solutions are valuable because:

- They can describe solutions where explicit solutions are difficult or impossible to obtain.

- They provide a complete relationship between variables.
- They facilitate understanding the geometric nature of solutions, such as level curves or solution contours.

Applications in Science and Engineering

Implicit solutions frequently appear in:

- Thermodynamics, where relationships like the ideal gas law are implicit.
- Electrical engineering, especially in circuit analysis involving nonlinear components.
- Population dynamics, where relationships between variables are inherently implicit.

Conclusion

The process of solving the differential equation $Y' + 2y = 7x$ with initial condition $Y(0) = 0$ demonstrates the power of integrating factors and integration by parts. The explicit solution:

$$y(x) = \frac{7x}{2} - \frac{7}{4} + \frac{7}{4} e^{-2x}$$

can be transformed into an implicit form:

$$4 y + 7 = 14 x + 7 e^{-2x},$$

which encapsulates the relationship between y and x. Recognizing and working with implicit solutions broadens our ability to analyze complex systems where explicit solutions might not be readily available. Whether in theoretical mathematics or practical applications, implicit solutions serve as an essential tool for understanding the intricate relationships governing dynamic systems.

Key Takeaways:

- The integrating factor method simplifies solving linear first-order differential equations.
- Initial conditions are crucial for determining particular solutions.
- Implicit solutions often provide a more comprehensive view of the relationship between variables.
- Transforming explicit solutions into implicit forms can reveal deeper insights into the solution structure.

By mastering these techniques, students and professionals can effectively analyze a wide range of differential equations and apply these methods to real-world problems across various scientific disciplines.

Frequently Asked Questions

What is the differential equation given in the problem?

The differential equation is $y' + 2y = 7x$ with the initial condition $y(0) = 0$.

How do you find the integrating factor for this differential equation?

The integrating factor is $e^{\int 2\, dx} = e^{2x}$.

What is the general solution to the differential equation y' + 2y = 7x?

The general solution is $y(x) = e^{-2x} (\int 7x\, e^{2x}\, dx + C)$.

How do you determine the particular solution using the initial condition y(0)=0?

Substitute $x=0$ and $y=0$ into the general solution to solve for C after integrating.

What is the implicit form of the solution after solving the integral?

The implicit solution is $4y + 7 = 14x + 7e^{-2x}$.

Additional Resources

Circle The Implicit Solution To The Differential Equation $Y' + 2y = 7x$, $Y(0) = 0$, A. $4y + 7 = 14x + 7e^{-2x}$

Introduction

Differential equations form the backbone of mathematical modeling across numerous scientific disciplines. Among them, first-order linear differential equations are fundamental, providing insight into systems where the rate of change of a quantity depends linearly on the quantity itself and an independent variable. The equation in focus, $Y' + 2Y = 7x$, embodies this class, and solving it explicitly or implicitly often reveals the dynamics underlying the modeled process.

In this article, we explore the solution to the initial value problem (IVP):

$$Y' + 2Y = 7x, \quad Y(0) = 0$$

and its connection to a related implicit form:

$$ 4Y + 7 = 14x + 7e^{-2x} $$

Our aim is to dissect the solution process thoroughly, understand the significance of the implicit form, and illustrate the geometric interpretation—particularly, how the solution curves relate to circles or other conic sections in the coordinate plane. This investigation combines classical techniques with modern insights, providing a comprehensive review suitable for researchers, students, and educators alike.

Background and Mathematical Foundations

The Nature of First-Order Linear Differential Equations

The general form of a first-order linear differential equation is:

$$ Y' + p(x)Y = q(x) $$

where $p(x)$ and $q(x)$ are functions of x. The standard approach involves finding an integrating factor:

$$ \mu(x) = e^{\int p(x) \, dx} $$

which transforms the equation into an exact derivative:

$$ \frac{d}{dx} [\mu(x)Y] = \mu(x)q(x) $$

and enables straightforward integration to find the general solution.

In our scenario:

$$ p(x) = 2 \quad \text{and} \quad q(x) = 7x $$

leading to an integrating factor:

$$ \mu(x) = e^{2x} $$

Connection to Implicit Solutions and Geometric Interpretation

While explicit solutions are often preferred, implicit solutions—expressed as relations involving x and

y without explicitly solving for y—can sometimes reveal geometric structures such as circles, ellipses, or hyperbolas. These structures provide deeper insight into the behavior of solutions over the domain and can be more informative in complex systems or when explicit solutions are cumbersome.

Solving the Differential Equation

Step 1: Find the Integrating Factor

Given:

$$Y' + 2Y = 7x$$

The integrating factor:

$$\mu(x) = e^{2x}$$

Step 2: Multiply the Equation by the Integrating Factor

$$e^{2x}Y' + 2e^{2x}Y = 7x e^{2x}$$

which simplifies to:

$$\frac{d}{dx} \left(e^{2x}Y \right) = 7x e^{2x}$$

Step 3: Integrate Both Sides

To find $Y(x)$:

$$e^{2x}Y = \int 7x e^{2x} dx + C$$

The integral on the right can be computed using integration by parts:

Let:

- $u = x \Rightarrow du = dx$
- $dv = e^{2x} dx \Rightarrow v = \frac{1}{2} e^{2x}$

Applying integration by parts:

$$\int x e^{2x} dx = x \cdot \frac{1}{2} e^{2x} - \int \frac{1}{2} e^{2x} dx = \frac{x}{2} e^{2x} - \frac{1}{4} e^{2x} + C'$$

Multiplying by 7:

$$ \int 7x e^{2x} dx = 7 \left(\frac{x}{2} e^{2x} - \frac{1}{4} e^{2x} \right) + C' = \frac{7x}{2} e^{2x} - \frac{7}{4} e^{2x} + C' $$

Step 4: Write the General Solution

Thus:

$$ e^{2x}Y = \frac{7x}{2} e^{2x} - \frac{7}{4} e^{2x} + C $$

Divide through by e^{2x}:

$$ Y = \frac{7x}{2} - \frac{7}{4} + Ce^{-2x} $$

Step 5: Apply Initial Condition $Y(0) = 0$

At $x=0$:

$$ 0 = \frac{7 \cdot 0}{2} - \frac{7}{4} + C e^{0} $$

$$ 0 = 0 - \frac{7}{4} + C $$

$$ C = \frac{7}{4} $$

Final Explicit Solution:

$$
\boxed{
Y(x) = \frac{7x}{2} - \frac{7}{4} + \frac{7}{4} e^{-2x}
}
$$

Deriving the Implicit Solution

Given the explicit form, we can manipulate to express an implicit relation:

$$
Y(x) = \frac{7x}{2} - \frac{7}{4} + \frac{7}{4} e^{-2x}
$$

Multiply both sides by 4:

$$
4Y = 14x - 7 + 7 e^{-2x}
$$

Rearranged as:

$$
4Y + 7 = 14x + 7 e^{-2x}
$$

This form:

$$
\boxed{
4Y + 7 = 14x + 7 e^{-2x}
}
$$

is the implicit relation provided in the initial problem statement.

Geometric Interpretation and the Circle Implication

Connecting the Implicit Solution to Circles

The implicit relation resembles a conic section, and the key is to interpret it geometrically. Consider the relation:

$$
4Y + 7 = 14x + 7 e^{-2x}
$$

which can be rearranged as:

$$
4Y = 14x + 7 e^{-2x} - 7
$$

or:

$$
Y = \frac{14x + 7 e^{-2x} - 7}{4}
$$

While the explicit form involves exponential and linear terms, the implicit form can be manipulated further to resemble the general form of a circle or other conic.

Approximating the Geometric Structure

Suppose we analyze the relation at specific points or limits:

- As $x \to 0$:

$$
e^{-2x} \to 1
$$

then,

$$
4Y + 7 \approx 14 \cdot 0 + 7 \cdot 1 = 7
$$

$$
\Rightarrow 4Y \approx 0 \Rightarrow Y \approx 0
$$

which is consistent with the initial condition.

- For large x:

The exponential term diminishes:

$$
e^{-2x} \to 0
$$

and the relation approaches:

$$
4Y + 7 \approx 14x
$$

$$
Y \approx \frac{14x - 7}{4}
$$

which is a straight line asymptote.

This asymptotic behavior suggests the solution curve approaches linearity at infinity but exhibits curvature influenced by the exponential term, hinting at a circle-like shape in the (x, Y) plane when viewed through appropriate transformations.

The Circle Analogy

While the explicit solution is not a perfect circle, the structure of the implicit relation and the exponential term suggest an implicit conic that can be transformed into a circle under certain substitutions.

Consider the exponential term:

$$
e^{-2x} = y
$$

which is a decaying exponential, and observe that in some contexts, relations involving exponentials and linear terms can be associated with circles or other conic sections in a higher-dimensional or parametric space.

Alternatively, by defining:

$$
z = e^{-2x}
$$

the relation becomes:

$$
4Y + 7 = 14x + 7z
$$

or:

$$
4Y = 14x + 7z - 7
$$

\]

which hints at a linear relation in \(x \) and \(z \), with \(z \) related exponentially to \(x \). This duality opens pathways to interpret the solution as a parametric curve, possibly inscribed within a circle or conic in a transformed space.

The Significance of the Implicit Solution and Its Geometric Implications

Why Implicit Solutions Matter

Implicit solutions are invaluable when explicit solutions are complicated or when the geometric nature of the solution curve is more transparent in a relation form. They can reveal:

- Underlying symmetries
- Conservation laws
- Geometric invariants

In our case, the relation hints at a geometric object akin to a circle

Circle The Implicit Solution To The Differential Equation Y 2y 7x Y 0 0 A 4y 7 14x 7e 2x

Circle The Implicit Solution To The Differential Equation Y 2y 7x Y 0 0 A 4y 7 14x 7e 2x

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