

Compute The Z-transforms Of The Following Signals. Cast Your Answer In The Form Of A Rational Fraction.

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Understanding the Z-transform is fundamental in discrete-time signal processing, control systems, and digital signal processing (DSP). It provides a powerful tool for analyzing linear, time-invariant systems, especially when dealing with difference equations and system stability. In this article, we will explore the process of computing the Z-transform for various signals, emphasizing how to express the result as a rational function of the complex variable z . Through detailed explanations and examples, you'll learn how to approach these calculations systematically.

Introduction to the Z-Transform

Before diving into calculations, it's essential to understand what the Z-transform is and why it's useful.

What Is the Z-Transform?

The Z-transform of a discrete-time signal $x[n]$ is defined as:

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

where:

- $x[n]$ is the discrete-time signal,
- z is a complex variable $z = re^{j\omega}$,
- The sum converges in a region called the Region of Convergence (ROC).

The Z-transform converts signals from the time domain into the complex frequency domain, simplifying the analysis of system behaviors such as stability and frequency response.

Why Express Z-Transform as a Rational Fraction?

Most signals' Z-transforms can be expressed as ratios of polynomials in z . These rational functions are advantageous because:

- They facilitate system analysis, especially in the context of transfer functions.
- They simplify the process of finding system stability and frequency characteristics.

- They make it easier to perform inverse Z-transform calculations.

Common Signals and Their Z-Transforms

Let's examine some typical signals and their Z-transforms, focusing on how to derive their rational function forms.

1. Unit Impulse Signal: $\delta[n]$

Definition:

$$\delta[n] = \begin{cases} 1, & n=0 \\ 0, & n \neq 0 \end{cases}$$

Z-Transform:

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = z^0 = 1$$

Result:

$$\boxed{X(z) = 1}$$

Expressed as a rational fraction, it is simply $1/1$.

2. Unit Step Signal: $u[n]$

Definition:

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

\]

Z-Transform:

$$X(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}} \quad \text{for } |z| > 1$$

Simplify:

$$X(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

Result:

$$\boxed{X(z) = \frac{z}{z - 1}}$$

This is a rational function in (z) with a pole at $(z=1)$.

3. Geometric Sequence: $(a^n u[n])$

Definition:

$$x[n] = a^n u[n], \quad |a| < \infty$$

Z-Transform:

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a z^{-1})^n = \frac{1}{1 - a z^{-1}}, \quad |a z^{-1}| < 1$$

Simplify:

$$X(z) = \frac{1}{1 - a z^{-1}} = \frac{z}{z - a}$$

Result:

$$\boxed{X(z) = \frac{z}{z - a}}$$

This is a rational function with a pole at $(z=a)$.

Calculating Z-Transforms for More Complex Signals

While basic signals are straightforward, more complex signals require systematic approaches, often involving properties like linearity, time-shifting, and convolution.

1. Time-Shifted Signals

Given $(x[n - n_0])$, the Z-transform shifts as:

$$\mathcal{Z}\{x[n - n_0]\} = z^{-n_0} X(z), \quad \text{assuming causality and ROC considerations}$$

Example:

$$x[n] = a^n u[n], \quad \text{then} \quad \mathcal{Z}\{x[n - 3]\} = z^{-3} \frac{z}{z - a}$$

2. Signal with Finite Duration

Suppose $(x[n] = A)$ for $(0 \leq n \leq N)$, zero elsewhere.

Z-Transform:

$$X(z) = \sum_{n=0}^N A z^{-n} = A \sum_{n=0}^N z^{-n}$$

which is a finite geometric series:

$$X(z) = A \frac{1 - z^{-(N+1)}}{1 - z^{-1}} = A \frac{z^{N+1} - 1}{z^{N+1}(z - 1)/z}$$

Alternatively, expressed as:

$$X(z) = A \frac{1 - z^{-(N+1)}}{1 - z^{-1}} = A \frac{z^{N+1} - 1}{z^N (z - 1)}$$

which is rational in z .

Handling Exponential and Sinusoidal Signals

Some signals involve exponential or sinusoidal components, which also have well-defined Z-transforms.

1. Sinusoidal Signal: $(\sin(\omega_0 n) u[n])$

The Z-transform of $(\sin(\omega_0 n) u[n])$ is:

$$X(z) = \frac{z \sin \omega_0}{z^2 - 2z \cos \omega_0 + 1}$$

This rational function in z reveals the pole-zero structure associated with sinusoidal signals.

2. Exponential Sinusoid: $(e^{\alpha n} \sin(\omega_0 n) u[n])$

The Z-transform is:

$$X(z) = \frac{z e^{j \omega_0} \sin \omega_0}{z^2 - 2 e^{\alpha} z \cos \omega_0 + e^{2 \alpha}}$$

Expressed as a rational function, it helps analyze damping and oscillation characteristics.

Practical Approach to Computing Z-Transforms

When computing the Z-transform of a given signal:

- 1. Identify the signal's form: Recognize if it is a basic sequence, shifted, scaled, or a combination.
- 2. Use known Z-transform pairs: Refer to standard tables of Z-transforms.
- 3. Apply linearity: Break complex signals into simpler components.
- 4. Use properties: Shifting, scaling, convolution, and differentiation can simplify calculations.
- 5. Express as a rational function: Simplify the sum or series into polynomial ratios.

Summary of Key Z-Transform Pairs

Signal	Z-Transform	ROC
$\delta[n]$	1	$ z > 0$
$u[n]$	$\frac{z}{z - 1}$	$ z > 1$
$a^n u[n]$	$\frac{z}{z - a}$	$ z > a $
Finite sequence $x[n]$	Polynomial/rational in z	depends on sequence

Conclusion

The process of computing Z-transforms and expressing them as rational fractions is fundamental in analyzing discrete-time systems. Recognizing standard sequences and applying properties like linearity, shifting, and convolution streamline the process. Expressing the Z-transform as a rational function in z not only simplifies analysis but also provides insights into system stability, pole-zero configuration, and frequency response. Whether dealing with basic signals like the unit step or more complex sequences, mastering these calculations enhances your ability to analyze and design digital systems effectively.

Remember: Always verify the ROC for convergence and stability considerations, especially when dealing with inverse Z-transforms or system analysis.

Further Reading and Resources:

Frequently Asked Questions

How do you compute the Z-transform of a discrete-time signal like $x[n] = a^n u[n]$ and express it as a rational function?

The Z-transform of $x[n] = a^n u[n]$ is $X(z) = 1 / (1 - a z^{-1})$, valid for $|z| > |a|$. To express as a rational fraction, rewrite as $X(z) = z / (z - a)$.

What is the Z-transform of a finite-duration signal, for example, $x[n] = n$ for $0 \leq n \leq N$ and zero elsewhere, in rational form?

The Z-transform is $X(z) = \sum_{n=0}^N n z^{-n}$. This sum can be expressed as a rational function using the formula for finite geometric series and its derivatives, resulting in a rational fraction in z .

How do you compute the Z-transform of a sinusoidal signal $x[n] = \sin(\omega_0 n) u[n]$, and express it as a rational function?

Express $\sin(\omega_0 n)$ in terms of complex exponentials: $(e^{j\omega_0 n} - e^{-j\omega_0 n}) / (2j)$. Then, the Z-transform becomes the difference of two geometric series, resulting in $X(z) = (z e^{j\omega_0} - z e^{-j\omega_0}) / (2j (z^2 - 2z \cos \omega_0 + 1))$, a rational function in z .

What is the Z-transform of a shifted impulse signal $x[n] = \delta[n - n_0]$, and how is it expressed as a rational fraction?

The Z-transform of $\delta[n - n_0]$ is $X(z) = z^{n_0}$. Since it's already a monomial in z , it is expressed as a rational function: $X(z) = z^{n_0}$.

How do you find the Z-transform of the exponential sequence $x[n] = a^n$ for $n \geq 0$ and express it as a rational function?

The Z-transform is $X(z) = 1 / (1 - a z^{-1}) = z / (z - a)$, valid for $|z| > |a|$. It is already expressed as a rational function in z .

When given a composite signal, such as $x[n] = \delta[n] + 2a^n u[n]$, how do you compute its Z-transform as a

rational fraction?

Compute the Z-transform of each component: $\delta[n]$ gives 1; $2a^n u[n]$ gives $2 / (1 - a z^{-1}) = 2z / (z - a)$. Sum them to get $X(z) = 1 + 2z / (z - a)$, which can be combined over a common denominator to form a rational fraction: $(z - a + 2z) / (z - a) = (3z - a) / (z - a)$.

Additional Resources

Compute The Z-transforms Of The Following Signals. Cast Your Answer In The Form Of A Rational Fraction.

Understanding the Z-transform is fundamental in the analysis and design of discrete-time systems, digital signal processing, and control systems. When asked to "Compute The Z-transforms of the following signals and cast your answer in the form of a rational fraction," we are essentially tasked with transforming a time-domain sequence into its complex frequency domain representation in a compact, algebraic form. This process simplifies convolution, stability analysis, and filter design, making the Z-transform a powerful tool for engineers and researchers alike.

In this comprehensive guide, we will walk through the fundamental concepts, step-by-step procedures, and practical examples to master the computation of Z-transforms and express them as rational functions. Whether you're a student tackling your first problems or a professional refining your understanding, this article aims to provide clarity and depth.

What is the Z-transform?

The Z-transform is a discrete-time counterpart to the Laplace transform, designed for sequences rather than continuous functions. It maps a discrete sequence $\{x[n]\}$ into a complex frequency domain $X(z)$, facilitating easier analysis of system properties like stability and frequency response.

Formal Definition

For a discrete sequence $\{x[n]\}$, the Z-transform is defined as:

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

where:

- z is a complex variable ($z = re^{j\omega}$),
- the sum converges within the Region of Convergence (ROC) in the complex plane.

Why Express Z-transforms as Rational Fractions?

Expressing Z-transforms as rational functions (ratios of polynomials in z) is crucial

because:

- It allows for straightforward analysis of system stability and causality.
- Polynomial ratios make filtering and system inversion easier.
- They facilitate partial fraction decomposition for inverse Z-transform calculations.
- Rational functions enable the application of standard Z-transform tables and properties.

General Approach to Computing Z-transforms

1. Identify the sequence $x[n]$: Whether it's a simple exponential, sinusoid, unit step, or a combination.
2. Use the Definition or Known Transforms: Leverage standard Z-transform pairs or derive from the definition.
3. Simplify the Sum or Expression: Apply algebraic manipulation, geometric series, or known identities.
4. Express as a Rational Function: Factor numerator and denominator polynomials in z .
5. Determine the Region of Convergence (ROC): Usually, for causal sequences, the ROC is outside the outermost pole; for anti-causal, inside the innermost pole.

Step-by-Step Examples

Example 1: Z-transform of a Causal Exponential Sequence

Sequence:

$$x[n] = a^n u[n], \quad |a| < \infty$$

where $u[n]$ is the unit step function.

Solution:

1. Write the Z-transform:

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n}$$

2. Recognize the series as geometric:

$$X(z) = \sum_{n=0}^{\infty} (a z^{-1})^n = \frac{1}{1 - a z^{-1}}, \quad \text{for } |a z^{-1}| < 1$$

$$|a z^{-1}| < 1$$

\]

3. Express as a rational function:

\[

$$X(z) = \frac{1}{1 - a z^{-1}} = \frac{z}{z - a}$$

\]

Final answer:

\[

\boxed{

$$X(z) = \frac{z}{z - a}$$

}

\]

with ROC: $|z| > |a|$.

Example 2: Z-transform of a Finite Sequence

Sequence:

\[

$$x[n] = \begin{cases}$$

$$1, \text{ \& } n = 0, 1, 2$$

$$0, \text{ \& } \text{otherwise}$$

$$\end{cases}$$

\]

Solution:

1. Write the sum:

\[

$$X(z) = \sum_{n=0}^2 x[n] z^{-n} = 1 \cdot z^0 + 1 \cdot z^{-1} + 1 \cdot z^{-2}$$

\]

2. Express as a rational function:

\[

$$X(z) = 1 + z^{-1} + z^{-2}$$

\]

3. Clear denominators:

\[

$$X(z) = \frac{z^2 + z + 1}{z^2}$$

\]

4. Express numerator as polynomial in z :

$$X(z) = \frac{z^2 + z + 1}{z^2}$$

Final answer:

$$\boxed{X(z) = \frac{z^2 + z + 1}{z^2}}$$

Advanced Techniques for Complex Sequences

1. Partial Fraction Expansion

When the Z-transform is given as a rational function, partial fractions facilitate the inverse transform:

- Decompose $X(z)$ into simpler fractions with linear or quadratic denominators.
- Use standard Z-transform pairs or tables for each term.

2. Using Known Pairs and Properties

Leverage known Z-transform pairs for common sequences, such as:

Sequence	Z-transform
$\delta[n]$	1
$u[n]$	$\frac{z}{z-1}$
$a^n u[n]$	$\frac{z}{z-a}$
$n a^n u[n]$	$\frac{a z}{(z-a)^2}$

and properties like shifting, scaling, and convolution.

Practical Tips for Computing Z-transforms

- Identify the sequence type: causal, anti-causal, finite-length, or infinite.
- Check convergence conditions: the ROC depends on causality.
- Use algebraic manipulation: factor polynomials, expand sums, or rewrite in geometric series.
- Apply known transforms: consult Z-transform tables or use software tools for complicated sequences.
- Express as rational fractions: always aim to write in polynomial numerator over polynomial denominator.

Summary: Key Takeaways

- The Z-transform converts discrete sequences into rational functions, simplifying analysis.
- Causal sequences often produce rational functions with ROC outside the outermost pole.
- Expressing the Z-transform in rational form involves geometric series and partial fraction decomposition.
- Mastery of standard pairs and properties accelerates the computation process.
- Rational form is essential for stability analysis, inverse transformation, and system design.

Final Remarks

Computing the Z-transforms and expressing them as rational fractions is both an art and a science. It requires understanding the underlying sequences, applying algebraic techniques, and leveraging known transforms. With practice, you'll be able to handle a wide variety of signals, enabling deeper insights into system behavior and signal processing tasks. Remember, the key is to always simplify your expressions into manageable polynomials and to understand the implications of the ROC for system stability and causality.

Happy transforming!

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