

Consider The Following Solid S. The Base Of S Is A Circular Disk With Radius R. Parallel Cross-sections

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Understanding the geometry of three-dimensional solids is a fundamental aspect of calculus and analytical geometry. Among the myriad of solid shapes, those constructed by stacking or slicing cross-sections are particularly interesting due to their versatility and applicability in real-world problems. In this article, we explore the properties, calculations, and applications of a specific class of solids—those with circular bases and parallel cross-sections.

Introduction to Solid S

Suppose we have a solid S whose base is a perfect circular disk with radius R. The base lies in the xy-plane, with its center at the origin (0,0). The key characteristic that distinguishes this solid is that its cross-sections taken parallel to the base are of a particular shape—often a circle, square, rectangle, or another geometric figure—that varies as we move along a certain axis.

Defining the Solid

- Base shape: Circular disk with radius R, lying in the xy-plane.
- Axis of interest: Usually the z-axis, perpendicular to the base.
- Cross-sections: Parallel to the base, taken at different heights z, which may be within the range $0 \leq z \leq H$, where H is the height of the solid.

The shape and size of the cross-sections as a function of z define the shape of the solid S.

Understanding Parallel Cross-sections

What Are Cross-sections?

A cross-section is the intersection of a solid with a plane. When the plane is parallel to a particular base, the cross-section provides a "slice" of the solid at a specific height. Analyzing these cross-sections is critical for calculating the volume and other properties of the solid.

Types of Cross-sections

Depending on the problem, the cross-sections might be:

- Circular: The cross-section is a circle at each height.
- Rectangular: The cross-section is a rectangle.
- Triangular: The cross-section forms a triangle.
- Other polygons: Such as squares, trapezoids, or more complex shapes.

In the context of a solid with a circular base, the shape of the cross-sections often varies systematically with height, either maintaining a consistent shape (e.g., always circular) or changing shape (e.g., from circle to square).

Mathematical Representation of Cross-sections

Circular Cross-sections

Suppose the cross-sections are circles parallel to the base. Then, at each height z , the radius of the cross-sectional circle, denoted $r(z)$, is a function of z .

- Example: If the solid tapers linearly from the base to the top, $r(z)$ might be proportional to the height:

$$r(z) = R \left(1 - \frac{z}{H} \right)$$

where H is the total height of the solid.

General Approach

To analyze such a solid, follow these steps:

1. Identify the shape of cross-sections: Determine the geometric shape at each z .
2. Express the dimensions of the cross-section as functions of z : For example, radius, side lengths, etc.
3. Set up integrals to compute volume: Integrate the area of the cross-section across the height z .

Calculating the Volume of Solid S

Method 1: Cross-sectional Method

The volume V of the solid is obtained by integrating the cross-sectional area $A(z)$:

$$V = \int_{z=0}^H A(z) \, dz$$

\]

where $A(z)$ is the area of the cross-section at height z .

Example: Circular Cross-sections

If the cross-sections are circles with radius $r(z)$:

$$A(z) = \pi [r(z)]^2$$

and the volume becomes:

$$V = \int_0^H \pi [r(z)]^2 \, dz$$

Depending on the specific form of $r(z)$, this integral can be evaluated explicitly.

Applications in Engineering and Science

Understanding solids with parallel cross-sections has numerous applications:

- Manufacturing: Designing objects with specific profiles, such as bottles, vases, or structural beams.
- Fluid dynamics: Calculating the volume of containers or flow channels.
- Physics: Analyzing moments of inertia for bodies with varying cross-sectional areas.
- Mathematics: Teaching integral calculus concepts through practical volume calculations.

Examples of Solid S with Different Cross-sectional Shapes

1. Cone with Circular Cross-sections

- Base: Circular disk with radius R .
- Shape: Linearly tapering from radius R at $z=0$ to 0 at $z=H$.
- Cross-sectional radius: $r(z) = R \left(1 - \frac{z}{H} \right)$.

Volume calculation:

$$V = \int_0^H \pi [r(z)]^2 \, dz = \pi R^2 \int_0^H \left(1 - \frac{z}{H} \right)^2 dz$$

which evaluates to:

$$V = \frac{1}{3} \pi R^2 H$$

matching the familiar formula for a cone.

2. Cylinder with Constant Cross-section

- Base: Circular disk with radius R .
- Shape: The cross-section remains a circle of radius R at every height z .

Volume:

$$V = A \times H = \pi R^2 H$$

This is the standard formula for a right circular cylinder.

3. Frustum of a Cone

- Shape: Truncated cone with larger radius R_1 at $z = 0$ and smaller radius R_2 at $z = H$.
- Cross-sectional radius: Varies linearly:

$$r(z) = R_1 + \left(\frac{R_2 - R_1}{H} \right) z$$

Volume:

$$V = \frac{\pi H}{3} \left(R_1^2 + R_1 R_2 + R_2^2 \right)$$

which is a common formula for the volume of a frustum.

Advanced Topics: Non-Uniform Cross-sections

While the above examples mainly focus on simple shapes, real-world applications often involve complex cross-sectional variations.

Non-Linear Variations

- The radius $r(z)$ may vary according to polynomial, exponential, or sinusoidal functions.

- Such variations require setting up the integral with the specific $r(z)$, which could involve advanced calculus techniques or numerical methods.

Non-Standard Shapes

- Cross-sections could be elliptical, rectangular, or irregular.
- Analyzing these involves integrating their respective areas as functions of z .

Visualizing the Solid

Creating accurate visualizations aids understanding:

- Sketch cross-sections: At various heights, to see how the shape changes.
- Use CAD software: To model and analyze complex solids with varying cross-sections.
- 3D plotting tools: Such as GeoGebra or MATLAB, facilitate comprehension of the shape.

Practical Steps for Analyzing Solid S

1. Define the base shape and dimensions.
2. Identify the axis of symmetry and the direction of cross-section variation.
3. Express the cross-sectional shape parameters as functions of the height z .
4. Write the formula for the area $A(z)$ of the cross-section.
5. Set up the integral for volume:

$$V = \int_{z=0}^H A(z) \, dz$$

6. Evaluate the integral analytically or numerically.

Conclusion

Analyzing solids with a circular base and parallel cross-sections is a cornerstone of integral calculus, providing insights into volume, surface area, and physical properties of three-dimensional objects. Recognizing how the cross-sectional shape and size vary with height allows for precise modeling and calculation, essential in engineering, physics, and mathematics.

By mastering the principles outlined in this article—such as defining the cross-sectional functions, setting up integrals, and interpreting the

results—students and professionals can approach complex geometric problems with confidence. Whether designing a tapered container, calculating the mass of a composite object, or developing structural components, understanding the interplay of base shape and cross-sectional variation is invaluable.

References

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- Thomas, G. B., & Finney, R. L. (2008). Calculus and Analytic Geometry. Addison-Wesley.

Note: The concepts discussed here extend to more complex geometries and can be adapted for computer-aided design, manufacturing, and scientific simulations.

Frequently Asked Questions

What is the general approach to finding the volume of a solid with a circular disk base and parallel cross-sections?

The general approach involves expressing the cross-sectional area as a function of the position along the axis and integrating this area across the entire height or length of the solid.

How do parallel cross-sections help in determining the volume of the solid S with a circular disk base?

Parallel cross-sections simplify volume calculation by breaking the solid into slices of known shape and area, which can then be integrated to find the total volume.

If the base of the solid S is a circle of radius R , what are common types of cross-sections used in volume calculations?

Common types include square, rectangle, equilateral triangle, or other similar shapes, taken perpendicular to a specified axis to generate the cross-sectional slices.

How does the shape of the cross-section influence the integral setup for calculating the volume?

The cross-sectional shape determines the expression for the area as a function of the position variable, which is integrated over the length or height to compute the volume.

Can you describe a typical problem involving parallel cross-sections of a solid with a circular base?

A typical problem might involve a solid with a circular base of radius R , where each cross-section is a square or triangle, and the task is to find the total volume by integrating the area of these cross-sections along the axis.

What is the significance of knowing the radius R of the base when analyzing the solid S with parallel cross-sections?

Knowing the radius R helps determine the maximum dimensions of the cross-sections at each point and is essential for expressing the cross-sectional area functions needed for integration.

Additional Resources

Consider The Following Solid S : The Base Of S Is A Circular Disk With Radius R . Parallel Cross-sections

Introduction

The study of three-dimensional solids is a fundamental aspect of calculus and geometry, offering insight into volume, surface area, and the behavior of shapes in space. Among various classes of solids, those with bases that are simple geometric figures—such as circles—are particularly instructive due to their symmetry and the ease with which cross-sectional areas can be described mathematically.

In this discussion, we focus on a specific solid, denoted as S , where the base is a circular disk of radius R . The defining characteristic of this solid is that its cross-sections, taken parallel to the base, have particular properties which influence its overall shape, volume, and surface area. The phrase “parallel cross-sections” suggests examining the shape of the solid at slices parallel to the base, which is a classic approach in calculus for volume determination.

This comprehensive review aims to dissect the geometry of such a solid in detail, exploring the implications of the shape of cross-sections, deriving formulas for volume, and understanding the various properties that arise from different conditions imposed on these cross-sections.

Geometric Description of the Solid S

Base Properties

- Shape: Circular disk
- Radius: (R)
- Position: Typically situated in the xy-plane, with the center at the origin or at some fixed point.

The base can be mathematically represented as:

$$\text{Base } B = \{(x, y) \mid x^2 + y^2 \leq R^2\}$$

This simple yet fundamental shape serves as the starting point for constructing the solid.

Vertical Extent and Symmetry

Depending on the specific properties of S, the solid may extend vertically to some height (h) , or may have a variable height depending on the position within the base. Symmetry plays a crucial role:

- If the solid is revolution-based, it may be symmetric about an axis.
- If the cross-sections are parallel to the base, the shape can vary significantly along the height.

Variations in Cross-Sectional Shapes

The key aspect of the problem lies in the nature of the cross-sections:

- Constant cross-sectional shape: e.g., all cross-sections are circles, squares, or rectangles.
- Variable cross-sectional shape: e.g., the cross-sections may change from circles to ellipses or other shapes as we move along the height.

In this review, we focus on the case where the cross-sections are parallel to the base, which implies they are taken in planes perpendicular to the vertical axis—assuming the solid extends vertically.

Parallel Cross-Sections: Fundamental Concepts

Definition and Importance

- Parallel cross-sections refer to slices of the solid made in planes parallel to the base.
- These sections provide a “slice-by-slice” perspective, allowing us to analyze and compute the volume by integrating the areas of each cross-section.

Mathematical Representation

Suppose the solid S extends along the (z) -axis from $(z=0)$ to $(z=h)$. At a height (z) , the cross-section is a shape whose area depends on (z) :

$$A(z) = \text{Area of the cross-section at height } z$$

The volume (V) of the solid can be obtained via the cylinder method:

$$V = \int_0^h A(z) \, dz$$

This integral approach provides a systematic way to calculate the volume when the cross-sectional shape is known or can be described mathematically.

Types of Cross-Sections and Their Implications

The shape and size of the cross-sections greatly influence the properties of the solid S . Several typical scenarios are considered:

1. Uniform Circular Cross-Sections

- Description: At every height (z) , the cross-section is a circle with radius $(r(z))$.
- Implication: The solid resembles a cylinder or a conical frustum, depending on how $(r(z))$ varies with (z) .
- Mathematical expression:

$$A(z) = \pi [r(z)]^2$$

- Volume calculation:

$$V = \int_0^h \pi [r(z)]^2 \, dz$$

- Special cases:

- Cylinder: $r(z) = R$ constant; volume = $(\pi R^2 h)$.

- Cone: $r(z) = \frac{R}{h}z$; volume = $(\frac{1}{3}\pi R^2 h)$.

2. Rectangular Cross-Sections

- Description: Each cross-section is a rectangle with width $w(z)$ and height $h(z)$.

- Implication: The volume is the integral of the product $w(z) \times h(z)$.

$$A(z) = w(z) \times h(z)$$

- Application: Used in modeling solids like prisms or more complex shapes.

3. Elliptical or Other Shapes

- Cross-sections could be ellipses with semi-axes $a(z)$ and $b(z)$:

$$A(z) = \pi a(z) b(z)$$

- More complex shapes can be analyzed similarly, provided their area formulas are known.

Case Study: The Solid S with Variable Cross-Sections

To deepen understanding, consider the scenario where the cross-sectional shape varies with height z , and the shape is always a circle with radius $r(z)$ that depends linearly or non-linearly on z .

Example 1: Conical Solid

- Description: The radius decreases linearly from R at $z=0$ to 0 at $z=h$:

$$r(z) = R \left(1 - \frac{z}{h}\right)$$

- Cross-sectional area:

$$A(z) = \pi \left(R \left(1 - \frac{z}{h}\right) \right)^2 = \pi R^2 \left(1 - \frac{z}{h}\right)^2$$

\]

- Volume:

\[

$$V = \int_0^h \pi R^2 \left(1 - \frac{z}{h}\right)^2 dz$$

\]

Calculating the integral:

\[

$$V = \pi R^2 \int_0^h \left(1 - \frac{z}{h}\right)^2 dz$$

\]

Substitute $(u = 1 - \frac{z}{h})$, $(du = -\frac{1}{h} dz)$, when $(z=0)$, $(u=1)$; when $(z=h)$, $(u=0)$:

\[

$$V = \pi R^2 h \int_0^1 u^2 du = \pi R^2 h \left[\frac{u^3}{3}\right]_0^1 = \frac{\pi R^2 h}{3}$$

\]

This matches the known volume of a cone, confirming the consistency of the approach.

Volume Calculation Techniques

1. Disk/Washer Method

This involves slicing the solid into thin disks or washers perpendicular to the axis:

- Advantages: Straightforward when the cross-sectional radius is known as a function of the height.
- Application: Used for solids like cones, cylinders, and other shapes with circular cross-sections.

2. Cross-Sectional Area Integration

- Directly integrating the cross-sectional area as a function of the variable:

\[

$$V = \int_a^b A(x) dx$$

\]

- This is especially useful when the cross-sectional area is easily expressed in terms of the variable of integration.

Surface Area Considerations

Beyond volume, the surface area of the solid S depends on the shape of its boundary. For solids with varying cross-sections:

- Lateral surface area can be computed by integrating the perimeter of the cross-section times the differential element along the axis:

$$A_{\text{lateral}} = \int_{z=0}^h P(z) \, dz$$

where $P(z)$ is the perimeter of the cross-section at height z .

- Total surface area includes the base, top (if closed), and lateral surfaces.

Applications and Examples

Engineering and Manufacturing

- Designing objects with specific volume and surface area requirements.
- Calculating the amount of material needed for manufacturing a conical or cylindrical object with variable cross-sections.

Physics

- Modeling the distribution of mass or charge in a 3D shape.
- Analyzing heat transfer or fluid flow in solids with such geometries.

Mathematics

- Teaching integral calculus through volume and surface area calculations.
- Exploring the properties of solids of revolution and more complex shapes.

Summary and Key Takeaways

- The shape and size of parallel cross-sections directly determine the volume and surface area of the solid.
- For solids with uniform cross-sections, volume calculation simplifies to area times height.
- When cross-sections vary, calculus provides the tools—integrating cross-sectional areas along the axis—to determine the volume.
- Recognizing the shape of the cross-section at each level helps in setting

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