Consider The Function F(x,y) = 8x3 + Y3 - 6xy + 2 A.) Find The Critical Points Of The Function. B.) Use

Consider The Function F(x,y) = 8x^3 + y^3 - 6xy + 2 A.) Find The Critical Points Of The Function. B.) Use

Understanding the behavior of functions of two variables is fundamental in multivariable calculus, especially when analyzing their critical points, which can indicate local maxima, minima, or saddle points. The function $f(x,y) = 8x^3 + y^3 - 6xy + 2$ presents an interesting case to study, as it combines cubic and linear terms, leading to complex behavior that warrants careful examination.

In this comprehensive guide, we will delve into the process of finding the critical points of the function $f(x,y)$, interpret what these points signify, and explore methods to classify them as maxima, minima, or saddle points. This approach is essential in various fields such as optimization, physics, economics, and engineering, where understanding the extrema of multivariable functions can inform decision-making and theoretical insights.

Understanding the Function $f(x,y) = 8x^3 + y^3 - 6xy + 2$

Before jumping into the calculations, it is helpful to understand the structure of the function:

- Cubic Terms: $8x^3$ and y^3 introduce non-linear behavior, with potential for multiple inflection points.
- Mixed Term: $-6xy$ couples the variables, influencing the shape and critical points.
- Constant Term: $+2$ shifts the entire function vertically.

This combination suggests the function has a rich landscape of peaks, valleys, and saddle points, typical for cubic functions with mixed variables.

Part A: Finding the Critical Points of the Function

Critical points occur where the gradient of the function equals zero. The gradient vector of $f(x,y)$ consists of the first-order partial derivatives:

$$
\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)
$$

Step 1: Compute the Partial Derivatives

- Partial derivative with respect to x:

$$
\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (8x^3 + y^3 - 6xy + 2) = 24x^2 - 6y
$$

- Partial derivative with respect to y:

$$
\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (8x^3 + y^3 - 6xy + 2) = 3y^2 - 6x
$$

Step 2: Set the Partial Derivatives Equal to Zero

To find critical points, solve the system:

$$
\begin{cases}
24x^2 - 6y = 0 \\
3y^2 - 6x = 0
\end{cases}
$$

Step 3: Solve the System of Equations

Rewrite the equations:

1. $24x^2 = 6y \Rightarrow y = 4x^2$

2. $3y^2 = 6x \Rightarrow y^2 = 2x$

Substitute $y = 4x^2$ into the second equation:

$$
(4x^2)^2 = 2x \Rightarrow 16x^4 = 2x
$$

Simplify:

$$
16x^4 - 2x = 0
$$

Factor:

$$
2x (8x^3 - 1) = 0
$$

Set each factor equal to zero:

- Factor 1: $2x = 0 \Rightarrow x = 0$

- Factor 2: $8x^3 - 1 = 0 \Rightarrow 8x^3 = 1 \Rightarrow x^3 = \frac{1}{8} \Rightarrow x = \left(\frac{1}{8}\right)^{1/3} = \frac{1}{2}$

Now, find corresponding y-values:

- For $x = 0$:

$$
y = 4(0)^2 = 0
$$

- For $x = \frac{1}{2}$:

\[

$$
y = 4 \left(\frac{1}{2}\right)^2 = 4 \times \frac{1}{4} = 1
$$

Critical Points of $F(x,y)$

The critical points are:

1. $\boxed{(0, 0)}$
2. $\boxed{\left(\frac{1}{2}, 1\right)}$

These points are candidates for local maxima, minima, or saddle points. To determine their nature, we need to analyze the second derivatives.

Part B: Classifying Critical Points Using the Second Derivative Test

The second derivative test for functions of two variables involves the Hessian matrix:

$$
H = \begin{bmatrix}
\frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\
\frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2}
\end{bmatrix}
$$

The nature of each critical point depends on the Hessian determinant D:

$$
D = \left(\frac{\partial^2 F}{\partial x^2} \right) \left(\frac{\partial^2 F}{\partial y^2} \right) - \left(\frac{\partial^2 F}{\partial x \partial y} \right)^2
$$

- If $D > 0$ and $\frac{\partial^2 F}{\partial x^2} > 0$, then a local minimum.
- If $D > 0$ and $\frac{\partial^2 F}{\partial x^2} < 0$, then a local maximum.
- If $D < 0$, then the critical point is a saddle point.
- If $D = 0$, the test is inconclusive.

Step 1: Compute Second Derivatives

- $\frac{\partial^2 F}{\partial x^2} = \frac{\partial}{\partial x} (24x^2 - 6y) = 48x$
- $\frac{\partial^2 F}{\partial y^2} = \frac{\partial}{\partial y} (3y^2 - 6x) = 6y$
- $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial}{\partial y} (24x^2 - 6y) = -6$

Step 2: Evaluate the Hessian and D at each critical point

At $(0, 0)$:

- $\frac{\partial^2 f}{\partial x^2} = 48 \times 0 = 0$
- $\frac{\partial^2 f}{\partial y^2} = 6 \times 0 = 0$
- $\frac{\partial^2 f}{\partial x \partial y} = -6$

Determinant:

$$
D = (0)(0) - (-6)^2 = -36 < 0
$$

Conclusion: The point $(0, 0)$ is a saddle point.

At $\left(\frac{1}{2}, 1\right)$:

- $\frac{\partial^2 f}{\partial x^2} = 48 \times \frac{1}{2} = 24 > 0$
- $\frac{\partial^2 f}{\partial y^2} = 6 \times 1 = 6 > 0$
- $\frac{\partial^2 f}{\partial x \partial y} = -6$

Determinant:

$$
D = (24)(6) - (-6)^2 = 144 - 36 = 108 > 0
$$

Since $\frac{\partial^2 f}{\partial x^2} > 0$ and $D > 0$, this point is a local minimum.

Summary of Critical Points and Their Classifications

Critical Point	Nature
(0, 0)	Saddle point
(1/2, 1)	Local minimum

Implications and Applications

Understanding the critical points of a function like $f(x,y)$ has practical significance in various disciplines:

- Optimization: Identifying minima and maxima helps in designing systems with optimal performance.
- Physics: Critical points correspond to equilibrium states in physical systems.
- Economics: Functions representing profit, cost, or utility often involve critical points indicating optimal or equilibrium scenarios.
- Engineering: Design parameters are optimized by analyzing multivariable functions for stability and efficiency.

Frequently Asked Questions

How do you find the critical points of the function $f(x, y) = 8x^3 + y^3 - 6xy + 2$?

To find the critical points, first compute the partial derivatives f_x and f_y, set them equal to zero, and solve the resulting system of equations. For this function, $f_x = 24x^2 - 6y$ and $f_y = 3y^2 - 6x$. Setting $f_x = 0$ gives $24x^2 = 6y$, so $y = 4x^2$. Setting $f_y = 0$ gives $3y^2 = 6x$, so $y^2 = 2x$. Substituting $y = 4x^2$ into $y^2 = 2x$ yields $(4x^2)^2 = 2x$ → $16x^4 = 2x$. Solving for x gives $x=0$ or $x=\pm(1/2)$. Plugging back into $y = 4x^2$ finds corresponding y-values, thus critical points are at $(0,0)$ and at $(\pm0.5, 4(\pm0.5)^2)$.

What method can be used to classify the critical points of the function $f(x, y) = 8x^3 + y^3 - 6xy + 2$?

The second derivative test using the Hessian matrix is used to classify critical points. Compute the second partial derivatives: f_{xx}, f_{yy}, and f_{xy}, then evaluate the determinant $D = f_{xx} f_{yy} - (f_{xy})^2$ at each critical point. If $D > 0$ and $f_{xx} > 0$, the point is a local minimum; if $D > 0$ and $f_{xx} < 0$, it's a local maximum; if $D < 0$, it's a saddle point.

How do you interpret the critical points found for the function $f(x, y) = 8x^3 + y^3 - 6xy + 2$?

Critical points indicate potential local maxima, minima, or saddle points. By analyzing the second derivatives at these points via the Hessian determinant, you can determine their nature. For example, if the Hessian determinant is positive and f_{xx} is positive, the point is a local minimum; if negative with f_{xx} negative, a local maximum; if negative determinant, a saddle point.

Can the method used to find and classify critical points be applied to other cubic functions, and what are the key steps?

Yes, the same method applies to other cubic functions. The key steps are: (1) find the partial derivatives with respect to x and y, (2) set these derivatives to zero to find critical points, (3) compute the second derivatives to form the Hessian matrix, (4) evaluate the Hessian determinant at each critical point, and (5) classify each point based on the sign of the determinant and second derivatives to determine if they are minima, maxima, or saddle points.

Additional Resources

Consider The Function $f(x,y) = 8x^3 + y^3 - 6xy + 2$

Introduction

Mathematics, especially multivariable calculus, provides powerful tools for analyzing complex functions that depend on multiple variables. One fundamental concept in this field is the identification of critical points, which serve as potential locations for local maxima, minima, or saddle points within a function. These points are essential in understanding the behavior of the function, optimizing certain values, or modeling real-world phenomena such as economics, physics, and engineering systems.

In this comprehensive review, we examine the function:

\[

$$
F(x, y) = 8x^3 + y^3 - 6xy + 2
$$

The goal is to determine its critical points, analyze their nature, and understand the implications of their locations and types. The process involves calculating the first and second derivatives, solving for points where derivatives vanish, and applying the second derivative test to classify the critical points.

Understanding the Function

Before delving into the calculus, it's important to grasp the structure of the given function.

The Components of $F(x,y)$

- Cubic terms: $8x^3$ and y^3 exhibit the typical growth of cubic functions, which are neither globally bounded nor symmetric.
- Mixed term: $-6xy$ introduces a coupling between x and y, influencing the shape and critical points' locations.
- Constant term: $+2$ shifts the entire surface vertically but does not affect the critical points' positions.

This mixture creates a rich landscape for analysis, with potential for multiple critical points, saddle points, and complex local behavior.

Part A: Finding the Critical Points of $F(x,y)$

Step 1: Recall the Definition of Critical Points

A critical point occurs where the gradient of the function vanishes. The gradient vector ∇F is composed of the partial derivatives with respect to each variable:

$$
\nabla F(x, y) = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right)
$$

Critical points satisfy:

$$
\frac{\partial F}{\partial x} = 0 \quad \text{and} \quad \frac{\partial F}{\partial y} = 0
$$

Step 2: Compute the Partial Derivatives

Let's differentiate $F(x, y)$:

- Partial derivative with respect to x:

$$
\frac{\partial F}{\partial x} = \frac{\partial}{\partial x} (8x^3 + y^3 - 6xy + 2) = 24x^2 - 6y
$$

- Partial derivative with respect to y:

$$
\frac{\partial F}{\partial y} = \frac{\partial}{\partial y} (8x^3 + y^3 - 6xy + 2) = 3y^2 - 6x
$$

Step 3: Set the Partial Derivatives Equal to Zero

We now solve the system:

$$
\begin{cases}
24x^2 - 6y = 0 \quad (1) \\
3y^2 - 6x = 0 \quad (2)
\end{cases}
$$

Step 4: Solve the System of Equations

Let's proceed step-by-step.

From equation (1):

$$
24x^2 = 6y \implies y = 4x^2
$$

Substitute into equation (2):

$$
3(4x^2)^2 - 6x = 0
$$

Simplify:

$$
3 \times 16x^4 - 6x = 0
$$

$$
48x^4 - 6x = 0
$$

Factor out $6x$:

$$
6x (8x^3 - 1) = 0
$$

Solutions:

- $6x = 0 \implies x = 0$

- $8x^3 - 1 = 0 \implies 8x^3 = 1 \implies x^3 = \frac{1}{8} \implies x = \left(\frac{1}{8}\right)^{1/3} = \frac{1}{2}$

Corresponding y values:

- For $x=0$:

$$
y = 4(0)^2 = 0
$$

- For $x = \frac{1}{2}$:

$$
y = 4 \left(\frac{1}{2}\right)^2 = 4 \times \frac{1}{4} = 1
$$

Critical Points Identified:

$$
\boxed{
\begin{cases}
(0, 0) \\
\left(\frac{1}{2}, 1\right)
\end{cases}
}
$$

Part B: Classifying the Critical Points

Identifying critical points is only part of the story. To fully understand their significance—whether they are maxima, minima, or saddle points—we need to analyze the second derivatives and apply the second derivative test.

Step 1: Compute Second Derivatives

Calculate the second partial derivatives:

$$
\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} (24x^2 - 6y) = 48x
$$

$$
\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} (3y^2 - 6x) = 6y
$$

$$
\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y} (24x^2 - 6y) = -6
$$

Note that mixed partial derivatives are equal ($\frac{\partial^2 f}{\partial y \partial x}$) by Clairaut's theorem, and both equal -6.

Step 2: Form the Hessian Determinant

The Hessian matrix H is:

$$
H = \begin{bmatrix}
\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\
\frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2}
\end{bmatrix}
= \begin{bmatrix}
48x & -6 \\
-6 & 6y
\end{bmatrix}
$$

\]

The determinant of the Hessian:

$$
D = (48x)(6y) - (-6)(-6) = 288xy - 36
$$

Step 3: Classify Each Critical Point Using the Second Derivative Test

The second derivative test states:

- If $D > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$, then f has a local minimum at the critical point.
- If $D > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0$, then f has a local maximum.
- If $D < 0$, then the critical point is a saddle point.
- If $D = 0$, the test is inconclusive.

Analyzing Critical Points

Critical Point 1: $(0, 0)$

- $\frac{\partial^2 f}{\partial x^2} = 48 \times 0 = 0$
- $\frac{\partial^2 f}{\partial y^2} = 6 \times 0 = 0$
- $D = 288 \times 0 \times 0 - 36 = -36 < 0$

Classification: Since $D < 0$, the point $(0, 0)$ is a saddle point.

Critical Point 2: $\left(\frac{1}{2}, 1\right)$

- $\frac{\partial^2 f}{\partial x^2} = 48 \times \frac{1}{2} = 24 > 0$
- $\frac{\partial^2 f}{\partial y^2} = 6 \times 1 = 6 > 0$
- $D = 288 \times \frac{1}{2} \times 1 - 36 = 288 \times 0.5 - 36 = 144 - 36 = 108 > 0$

Since $D > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$, the point $\left(\frac{1}{2}, 1\right)$ is a local minimum.

Visualizing the Function's Surface

Understanding these critical points' nature illuminates the overall shape of the surface defined by $f(x, y)$:

- The saddle point at $(0, 0)$

Consider The Function F X Y 8x3 Y3 6xy 2 A Find The Critical Points Of The Function B Use

Consider The Function F X Y 8x3 Y3 6xy 2 A Find The Critical Points Of The Function B Use

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