Consider The System Of Differential Equations X_1(t) = -1x_1+0X_2x_2(t) = -12x_1+-7x_2where X_1 And X_2

Consider The System Of Differential Equations X_1(t) = -1x_1 + 0X_2, X_2(t) = -12x_1 - 7x_2 where X_1 and X_2

Understanding systems of differential equations is fundamental in various scientific and engineering disciplines, including physics, biology, economics, and control systems. The system described by the equations:

\[
\begin{cases}
\frac{dX_1}{dt} = -1 \times X_1 + 0 \times X_2 \\
\frac{dX_2}{dt} = -12 \times X_1 - 7 \times X_2
\end{cases}
\]

serves as a classic example to analyze coupled differential equations, eigenvalues, eigenvectors, and stability of the system. This article provides a comprehensive exploration of this system, including its formulation, solution methods, interpretation of results, and applications.

Understanding the System of Differential Equations

What Are Differential Equations?

Differential equations are mathematical equations that relate a function to its derivatives. They describe how a quantity changes over time or space. In systems involving multiple variables, such as the one under consideration, coupled differential equations describe how variables influence each other dynamically.

The Given System Explained

The system:

\[
\begin{cases}
\frac{dX_1}{dt} = -X_1 \\
\frac{dX_2}{dt} = -12X_1 - 7X_2
\end{cases}

\]

can be represented in matrix form as:

$$
\frac{d\mathbf{X}}{dt} = A \mathbf{X}
$$

where

$$
\mathbf{X} = \begin{bmatrix}
X_1(t) \\
X_2(t)
\end{bmatrix},
\quad
A = \begin{bmatrix}
-1 & 0 \\
-12 & -7
\end{bmatrix}
$$

This formulation simplifies the analysis, enabling us to utilize matrix algebra, eigenvalues, and eigenvectors to find solutions.

Matrix Representation and Its Significance

Why Use Matrix Form?

Transforming the differential equations into matrix form allows for straightforward application of linear algebra techniques. It enables:

- Finding eigenvalues and eigenvectors for system behavior analysis.
- Simplifying the process of solving coupled equations.
- Understanding the stability and nature of solutions.

Matrix Components in Our System

The matrix A encapsulates the interactions:

- The diagonal elements (-1 and -7) represent how each variable influences its own rate of change.
- The off-diagonal element -12 indicates how X_1 influences X_2.

Solving the System of Differential Equations

Step 1: Find Eigenvalues of Matrix A

Eigenvalues determine the nature of the solutions — whether they decay, grow, or oscillate.

To find eigenvalues λ, solve:

$$
\det(A - \lambda I) = 0
$$

where I is the identity matrix.

Compute:

$$
\det \begin{bmatrix}
-1 - \lambda & 0 \\
-12 & -7 - \lambda
\end{bmatrix} = 0
$$

$$
(-1 - \lambda)(-7 - \lambda) - (0)(-12) = 0
$$

$$
(-1 - \lambda)(-7 - \lambda) = 0
$$

Expanding:

$$
(-1)(-7) + (-1)(-\lambda) + (-\lambda)(-7) + \lambda^2 = 0
$$

$$
7 + \lambda + 7\lambda + \lambda^2 = 0
$$

$$
\lambda^2 + 8\lambda + 7 = 0
$$

Solve the quadratic:

$$
\lambda = \frac{-8 \pm \sqrt{64 - 28}}{2} = \frac{-8 \pm \sqrt{36}}{2}
$$

$$
\lambda = \frac{-8 \pm 6}{2}
$$

Thus,

$$
\boxed{
\lambda_1 = \frac{-8 + 6}{2} = -1, \quad \lambda_2 = \frac{-8 - 6}{2} = -7
}
$$

Step 2: Find Eigenvectors Corresponding to Eigenvalues

Eigenvector for $\lambda_1 = -1$:

Solve:

$$
(A - (-1)I)\mathbf{v} = 0
$$

$$
\begin{bmatrix}
-1 + 1 & 0 \\
-12 & -7 + 1
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
= 0
\Rightarrow
\begin{bmatrix}
0 & 0 \\
-12 & -6
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
= 0
$$

The second row:

$$
-12 v_1 - 6 v_2 = 0 \Rightarrow 2 v_1 + v_2 = 0
$$

Choose $v_1 = 1$, then $v_2 = -2$.

Eigenvector:

$$
\mathbf{v}_1 = \begin{bmatrix}
1 \\
-2
\end{bmatrix}
$$

Eigenvector for $\lambda_2 = -7$:

Solve:

$$
A - (-7)I = \begin{bmatrix}
-1 + 7 & 0 \\
-12 & -7 + 7
\end{bmatrix} = \begin{bmatrix}
6 & 0 \\
-12 & 0
\end{bmatrix}
$$

The second row:

$$
-12 v_1 + 0 v_2 = 0 \Rightarrow v_1 = 0
$$

From the first row:

$$
6 v_1 + 0 v_2 = 0 \Rightarrow v_1 = 0
$$

v_2 is arbitrary; choose $v_2 = 1$.

Eigenvector:

$$
\mathbf{v}_2 = \begin{bmatrix}
0 \\
1
$$

\end{bmatrix}
\]

Step 3: General Solution to the System

The general solution:

$$
\mathbf{X}(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2
$$

which explicitly is:

$$
\mathbf{X}(t) = c_1 e^{-1 t} \begin{bmatrix}
1 \\
-2
\end{bmatrix} + c_2 e^{-7 t} \begin{bmatrix}
0 \\
1
\end{bmatrix}
$$

or in component form:

$$
X_1(t) = c_1 e^{-t}
$$
$$
X_2(t) = -2 c_1 e^{-t} + c_2 e^{-7 t}
$$

Constants c_1, c_2 are determined by initial conditions.

Interpreting the Solution and System Dynamics

Stability Analysis

Since both eigenvalues -1 and -7 are negative, solutions decay exponentially over time, indicating the system is stable and tends to equilibrium at $(0,0)$.

Behavior of the Variables

- $X_1(t)$: Decays exponentially with a rate of e^{-t}.
- $X_2(t)$: Also decays exponentially, influenced both by the decay of $X_1(t)$ and by its own decay rate $e^{-7 t}$.

The system exhibits a combination of fast and slow decay modes, with the X_2 variable having a component that decays more rapidly due to the $e^{-7 t}$ term.

Physical and Practical Implications

- The zero coefficient in the first differential equation indicates X_1 evolves independently, decaying exponentially.
- The second equation shows X_2 depends on X_1, highlighting a coupling where X_1 influences X_2's behavior.
- In real-world systems, such dynamics could represent coupled decay processes, such as in chemical reactions, population dynamics, or electrical circuits.

Applications of Such Differential Equations Systems

Engineering and Control Systems

- Designing controllers that stabilize system variables.
- Analyzing feedback loops where variables influence each other.

Physics and Chemistry

- Modeling radioactive decay chains.
- Understanding coupled oscillations and damping.

Biological

Frequently Asked Questions

What is the general form of the system of differential equations given by $X_1(t) = -1X_1 + 0X_2$ and $X_2(t) = -12X_1 - 7X_2$?

The system represents two coupled linear differential equations: $X_1' = -X_1$ and $X_2' = -12X_1 - 7X_2$, which can be written in matrix form as $X' = AX$, where A is a 2x2 matrix with entries [-1, 0; -12, -7].

How do you find the eigenvalues of the coefficient matrix for this system?

Eigenvalues are found by solving the characteristic equation $\det(A - \lambda I) = 0$, where A is the coefficient matrix. For this system, the matrix is [[-1, 0], [-12, -7]], so the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -7$.

What are the general solutions for $X_1(t)$ and $X_2(t)$ in this system?

The general solutions are $X_1(t) = C_1e^{-t}$ and $X_2(t) = C_2e^{-7t}$ + (additional particular solution if the system is nonhomogeneous). Since the equations are coupled, $X_2(t)$ can be expressed in terms of $X_1(t)$ using the eigenvalues and eigenvectors.

Are the solutions to this system stable, and why?

Yes, the solutions are stable because both eigenvalues have negative real parts (-1 and -7), which means the system's trajectories decay exponentially to zero over time.

How can the initial conditions influence the particular solution of this system?

Initial conditions determine the constants C_1 and C_2 in the general solution, thus specifying the exact trajectory of the system's evolution over time.

Can this system be decoupled to solve for X_1 and X_2 separately?

Yes, since $X_1' = -X_1$ is independent of X_2, it can be solved directly. Once $X_1(t)$ is known, it can be substituted into the second equation to solve for $X_2(t)$.

What is the significance of the zero coefficient in the first equation, and how does it affect the system?

The zero coefficient indicates that X_1's rate of change depends only on X_1 itself, making its solution straightforward. It also means X_2 does not influence X_1 directly, simplifying the analysis of the system.

Additional Resources

Consider The System Of Differential Equations X_1(t) = -1x_1+0X_2x_2(t) = -12x_1+-7x_2

Introduction

The study of systems of differential equations is fundamental in understanding the dynamics of complex phenomena across disciplines such as physics, biology, engineering, and economics. These systems provide a mathematical framework for modeling how multiple interdependent variables evolve over time. Among these, linear systems with constant coefficients are particularly well-understood due to their analytical tractability and wide applicability.

In this review, we focus on a specific two-dimensional linear system characterized by the equations:

$$
\begin{cases}
X_1'(t) = -1 \times X_1(t) + 0 \times X_2(t) \\
X_2'(t) = -12 \times X_1(t) - 7 \times X_2(t)
\end{cases}
$$

which can be succinctly written as:

\[
\frac{d}{dt}
\begin{bmatrix}
X_1(t) \\
X_2(t)
\end{bmatrix}
=
A
\begin{bmatrix}
X_1(t) \\
X_2(t)

\end{bmatrix}
\]

where the coefficient matrix \(A\) is:

\[
A = \begin{bmatrix}
-1 & 0 \\
-12 & -7
\end{bmatrix}
\]

This system encapsulates a range of dynamical behaviors, including stability, oscillation, and exponential decay, depending on the eigenstructure of \(A\). Our examination will delve into the analytical solution, eigenvalue analysis, phase portrait interpretation, and potential applications of such a system.

Mathematical Formulation and Basic Properties

System Representation

The system under consideration is:

\[
\frac{d\mathbf{X}}{dt} = A\mathbf{X}
\]

where

\[

$$
\mathbf{X}(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix}
$$

and

$$
A = \begin{bmatrix}
-1 & 0 \\
-12 & -7
\end{bmatrix}
$$

This representation indicates a linear, autonomous system with constant coefficients.

Initial Conditions and Solution Strategy

To analyze solutions, an initial condition vector:

$$
\mathbf{X}(0) = \begin{bmatrix} X_1(0) \\ X_2(0) \end{bmatrix}
$$

must be specified. The goal is to find $\mathbf{X}(t)$ satisfying:

$$
\mathbf{X}(t) = e^{At} \mathbf{X}(0)
$$

where e^{At} is the matrix exponential.

Eigenvalue Analysis and System Behavior

Eigenvalues and Eigenvectors

The core step in understanding the system's qualitative behavior involves computing the eigenvalues λ of matrix A:

$$
\det(A - \lambda I) = 0
$$

which yields the characteristic equation:

$$
\det
\begin{bmatrix}
-1 - \lambda & 0 \\
-12 & -7 - \lambda
\end{bmatrix} = 0
$$

Calculating the determinant:

$$
(-1 - \lambda)(-7 - \lambda) - (0)(-12) = 0
$$

$$
(-1 - \lambda)(-7 - \lambda) = 0
$$

Expanding:

$$
(-1)(-7) + (-1)(-\lambda) + (-\lambda)(-7) + \lambda^2 = 0
$$

$$
7 + \lambda + 7 \lambda + \lambda^2 = 0
$$

$$
\lambda^2 + 8\lambda + 7 = 0
$$

Solving the quadratic:

$$
\lambda = \frac{-8 \pm \sqrt{64 - 28}}{2} = \frac{-8 \pm \sqrt{36}}{2} = \frac{-8 \pm 6}{2}
$$

Thus,

$$
\boxed{
\lambda_1 = \frac{-8 + 6}{2} = -1, \quad
\lambda_2 = \frac{-8 - 6}{2} = -7
}
$$

Eigenvalues Summary:
- $\lambda_1 = -1$
- $\lambda_2 = -7$

Both eigenvalues are real and negative, indicating the system's solutions decay exponentially to equilibrium, with the rate determined by the eigenvalues.

Eigenvectors

Eigenvector for $\lambda_1 = -1$:

Solve:

$$
(A + I) \mathbf{v} = 0
$$

$$
\begin{bmatrix}
-1 + 1 & 0 \\
-12 & -7 + 1
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
=
\begin{bmatrix}
0 & 0 \\
-12 & -6
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
$$

= 0
\]

From the second row:

\[
-12 v_1 - 6 v_2 = 0 \Rightarrow 2 v_1 + v_2 = 0 \Rightarrow v_2
= -2 v_1
\]

Choosing \(v_1 = 1\):

\[
\boxed{
\mathbf{v}^{(1)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}
}
\]

Eigenvector for \(\lambda_2 = -7\):

Solve:

\[
(A + 7 I) \mathbf{v} = 0
\]

\[
\begin{bmatrix}
-1 + 7 & 0 \\
-12 & -7 + 7
\end{bmatrix}
= \begin{bmatrix}
6 & 0 \\
-12 & 0

\end{bmatrix}
\]

Second row:

\[
-12 v_1 = 0 \Rightarrow v_1 = 0
\]

First row:

\[
6 v_1 = 0
\]

which is satisfied for any \(v_2\). Choosing \(v_2 = 1\):

\[
\boxed{
\mathbf{v}^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
}
\]

Analytical Solution of the System

Given the eigenvalues and eigenvectors, the general solution can be expressed as a linear combination:

\[
\mathbf{X}(t) = c_1 e^{\lambda_1 t} \mathbf{v}^{(1)} + c_2 e^{\lambda_2 t} \mathbf{v}^{(2)}
\]

which explicitly is:

$$
\mathbf{X}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{-7t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}
$$

or component-wise:

$$
X_1(t) = c_1 e^{-t}
$$

$$
X_2(t) = -2 c_1 e^{-t} + c_2 e^{-7t}
$$

The constants c_1, c_2 are determined by initial conditions:

$$
\mathbf{X}(0) = \begin{bmatrix} X_1(0) \\ X_2(0) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}
$$

which yields:

$$
X_1(0) = c_1
$$

$$
X_2(0) = -2 c_1 + c_2
$$

\]

Thus,

\[
c_1 = X_1(0)
\]

\[
c_2 = X_2(0) + 2 X_1(0)
\]

Qualitative Dynamics and Phase Portraits

Decay to Equilibrium

Because both eigenvalues are negative, solutions tend toward the equilibrium point at the origin:

\[
\mathbf{X}^ = \mathbf{0}
\]

This indicates the system is asymptotically stable, with trajectories decaying exponentially toward zero.

Trajectory Behavior

The decoupling of the solutions into exponential modes implies the phase portrait comprises trajectories that exponentially approach the origin along directions dictated by the eigenvectors.

- The mode associated with $\lambda_1 = -1$ decays slower than $\lambda_2 = -7$, meaning the long-term behavior is dominated by the eigenvector $\mathbf{v}^{(1)}$.
- The initial condition's projection onto these eigenvectors determines the trajectory shape.

Visualizing the System

Plotting trajectories in the X_1-X_2 plane reveals:

- All trajectories spiral inward along straight lines

[Consider The System Of Differential Equations X 1 T 1x 1 0x 2x 2 T 12x 1 7x 2where X 1 And X 2]()

Consider The System Of Differential Equations X 1 T 1x 1 0x 2x 2 T 12x 1 7x 2where X 1 And X 2

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