Construct A Polynomial Function With The Following Properties Fifth Degree, 3 Is A Zero Of Multiplicity

Construct A Polynomial Function With The Following Properties: Fifth Degree, 3 Is A Zero Of Multiplicity

Creating polynomial functions with specific properties is a fundamental skill in algebra and calculus, especially for students and professionals working in mathematical modeling, engineering, and sciences. When asked to construct a polynomial function that is of fifth degree and has a zero at 3 with a certain multiplicity, it involves understanding the factors, zeros, and the degree of the polynomial, as well as how multiplicity affects the graph's behavior. In this comprehensive guide, we will walk through the process of constructing such a polynomial, explore key concepts involved, and provide practical examples to solidify understanding.

Understanding the Key Concepts

Before diving into the construction process, it is essential to clarify the primary concepts involved in the problem.

Degree of a Polynomial

- The degree of a polynomial is the highest power of the variable x with a non-zero coefficient.
- For a fifth-degree polynomial, the highest exponent of x is 5.
- The degree determines the end behavior of the polynomial's graph and influences the number of roots (including multiplicities).

Zeros (Roots) of a Polynomial

- Zeros are the values of x where the polynomial equals zero.
- They are related to the factors of the polynomial; if a is a zero, then $(x - a)$ is a factor.
- The multiplicity of a zero indicates how many times that zero occurs as a root.

Multiplicity of a Zero

- The multiplicity of a zero refers to the number of times a particular zero appears.
- For example, if $(x - 3)^2$ is a factor, then zero $x=3$ has multiplicity 2.
- The multiplicity affects the shape of the graph at the zero:
- Even multiplicity (like 2, 4): the graph touches the x-axis and turns around.
- Odd multiplicity (like 1, 3): the graph crosses the x-axis.

Constructing the Polynomial

- To construct a polynomial with given zeros and multiplicities, form factors accordingly.
- Multiply the factors and expand if necessary.
- Adjust the leading coefficient to meet additional criteria if specified.

Step-by-Step Guide to Constructing the Polynomial

Constructing a polynomial with the specified properties involves several steps.

Step 1: Identify the Zero and Its Multiplicity

- Zero: $x = 3$
- Since the polynomial is fifth degree, and the zero at 3 has a certain multiplicity, ensure the total degree sums to 5.

Step 2: Decide the Multiplicity of Zero at 3

- The multiplicity of zero at 3 can be 1, 2, 3, 4, or 5.
- The sum of multiplicities across all zeros must be 5.
- For simplicity, assume zero at 3 has multiplicity 3, which accounts for degree 3.

Step 3: Determine Additional Zeros to Reach Fifth Degree

- Since the total degree is 5, and zero at 3 has multiplicity 3, the remaining degree is 2.
- To reach degree 5, introduce additional zeros with their multiplicities summing to 2.

Possible options include:

- Zero at $x = a$ with multiplicity 2.
- Zero at $x = b$ with multiplicity 2.
- Zeros at different points with multiplicities adding up to 2, e.g., 1 and 1.

For simplicity, assume:

- Zero at $x = 3$ with multiplicity 3.
- Zero at $x = 2$ with multiplicity 2.

Constructing the Polynomial

Based on the above, the polynomial factors are:

$$
f(x) = k \times (x - 3)^3 \times (x - 2)^2
$$

where k is a non-zero constant that can be chosen as 1 for simplicity unless specified otherwise.

Step 4: Write the Polynomial in Factored Form

$$
f(x) = (x - 3)^3 \times (x - 2)^2
$$

This polynomial is:

- Fifth degree (degree 3 + 2).
- Has zero at $x=3$ with multiplicity 3.
- Has zero at $x=2$ with multiplicity 2.

Step 5: Expand the Polynomial (Optional)

If you need the polynomial in standard form:

1. Expand \((x - 3)^3 \):

\[
(x - 3)^3 = x^3 - 9x^2 + 27x - 27
\]

2. Expand \((x - 2)^2 \):

\[
(x - 2)^2 = x^2 - 4x + 4
\]

3. Multiply the two:

\[
f(x) = (x^3 - 9x^2 + 27x - 27)(x^2 - 4x + 4)
\]

Applying distributive property (FOIL):

\[
f(x) = x^3 \times (x^2 - 4x + 4) - 9x^2 \times (x^2 - 4x + 4) + 27x \times (x^2 - 4x + 4) - 27 \times (x^2 - 4x + 4)
\]

Calculations:

- \(x^3 \times x^2 = x^5 \)
- \(x^3 \times -4x = -4x^4 \)
- \(x^3 \times 4 = 4x^3 \)

- \(-9x^2 \times x^2 = -9x^4 \)
- \(-9x^2 \times -4x = 36x^3 \)
- \(-9x^2 \times 4 = -36x^2 \)

- \(27x \times x^2 = 27x^3 \)
- \(27x \times -4x = -108x^2 \)
- \(27x \times 4 = 108x \)

- \(-27 \times x^2 = -27x^2 \)
- \(-27 \times -4x = 108x \)
- \(-27 \times 4 = -108 \)

Now, combine like terms:

\[

$$
f(x) = x^5 + (-4x^4 - 9x^4) + (4x^3 + 36x^3 + 27x^3) + (-36x^2 - 108x^2 - 27x^2) + (108x + 108x) - 108
$$

Simplify:

$$
f(x) = x^5 - 13x^4 + 67x^3 - 171x^2 + 216x - 108
$$

This is the expanded form of the polynomial.

Additional Considerations and Variations

While the above example constructs a specific polynomial, there are various ways to tailor the polynomial to meet different criteria.

Adjusting the Leading Coefficient

- If you need the polynomial to be monic (leading coefficient 1), set $k=1$.
- To scale the polynomial, choose k accordingly.

Choosing Different Zeros

- The zeros can be any real or complex numbers.
- The multiplicities can vary, provided the sum equals 5.
- For example, zeros at $x = -1$ and $x=4$ with multiplicities 2 and 3 respectively.

Ensuring Polynomial Behavior

- The multiplicity at each zero affects the graph's shape.
- Even multiplicities cause the graph to touch the x-axis and turn around.
- Odd multiplicities cause the graph to cross the x-axis.

Constructing with Complex Zeros

- Complex zeros come in conjugate pairs.

- For instance, zeros at $2 + i$ and $2 - i$, each with multiplicity 1, contribute to real polynomials.

Practical Applications and Examples

Constructing polynomials with specific zeros and multiplicities is useful in various real-world applications:

- Engineering: Designing control systems with specific response characteristics.
- Physics: Modeling phenomena with roots at certain points due to boundary conditions.
- Mathematics: Analyzing polynomial behavior and graphing functions with specified properties.

Example 1: Construct a fifth-degree polynomial with zero at $x=3$ of multiplicity 3, and zeros at $x=0$ and $x=4$ of multiplicity 1 each.

Solution:

$$
f(x) = (x - 3)^3 \times x \times (x - 4)
$$

Expanded:

- $(x - 3)^3 = x^3 - 9x^2 + 27x$

Frequently Asked Questions

How do you construct a fifth-degree polynomial function with a zero at 3 of multiplicity 2?

Start with factors like $(x - 3)^2$ to account for the zero at 3 with multiplicity 2, then include additional factors to reach degree 5, such as $(x - a)^1$, $(x - b)^1$, and $(x - c)^1$, where a, b, c are other zeros. Multiply all factors together to form the polynomial.

What does it mean for 3 to be a zero of multiplicity 3 in a fifth-degree polynomial?

It means the factor (x - 3) appears three times in the polynomial's factorization, contributing a zero at x=3 with multiplicity 3, and the overall degree of the polynomial is 5.

Can a fifth-degree polynomial have a zero at 3 with multiplicity 1 and still satisfy certain properties?

Yes, a fifth-degree polynomial can have a zero at 3 with multiplicity 1, but to meet specific properties—such as the zero at 3 having a certain multiplicity—you must include the factor (x - 3) exactly that many times during construction.

What steps are involved in constructing a polynomial of degree 5 with a zero at 3 of multiplicity 2?

First, include the factor (x - 3)^2 to ensure the zero at 3 has multiplicity 2. Then, choose three additional factors corresponding to other zeros to reach degree 5, multiply all factors, and expand to obtain the polynomial.

How does the multiplicity of a zero affect the graph of a fifth-degree polynomial?

The multiplicity determines how the graph touches or crosses the x-axis at that zero. For example, a zero with odd multiplicity (like 3) causes the graph to cross the x-axis at that point, while even multiplicity causes it to touch and bounce off the axis.

If a fifth-degree polynomial has a zero at 3 with multiplicity 3, what is the general form of the polynomial?

The general form includes the factor (x - 3)^3 multiplied by a quadratic factor, such as (x - a)(x - b), to ensure the degree is 5. So, it looks like P(x) = k(x - 3)^3(x - a)(x - b), where k is a nonzero constant.

Additional Resources

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Introduction

Polynomial functions are fundamental constructs in algebra, serving as building blocks for numerous applications across mathematics, physics, engineering, and computer science. Their properties—such as degree, roots, and multiplicities—offer insight into their behavior, shape, and applications. Among the vast landscape of polynomial functions, those of specific degrees with prescribed roots and multiplicities present intriguing challenges and opportunities for exploration.

In particular, constructing a fifth-degree polynomial where 3 is a zero of multiplicity is a compelling problem that combines the fundamental concepts of polynomial degrees, roots, and multiplicities. This investigation delves into the theoretical underpinnings, practical methods, and nuanced considerations involved in designing such functions, providing a comprehensive guide for educators, students, and researchers alike.

Understanding the Foundation: Polynomial Degree, Roots, and Multiplicity

What Is a Polynomial Function?

A polynomial function $P(x)$ of degree n can be expressed as:

$$
P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0
$$

where:

- $a_n \neq 0$,
- n is a non-negative integer indicating the degree.

The fundamental theorem of algebra states that a polynomial of degree n has exactly n roots (counting multiplicities) in the complex plane.

Roots and Multiplicity: Key Concepts

- Root (Zero): A value r such that $P(r) = 0$.
- Multiplicity: The number of times a root appears as a factor in the polynomial's factorization. For example, if $(x - 3)^k$ divides $P(x)$, then 3 is a root of multiplicity k.

The multiplicity influences the shape of the graph at the root:

- Odd multiplicity: The graph crosses the x-axis at the root.
- Even multiplicity: The graph touches the x-axis but does not cross it.

The Core Problem: Constructing a Fifth-Degree Polynomial with a Zero of Multiplicity at 3

Defining the Parameters

Given the problem statement, the goal is to construct a polynomial $P(x)$ such that:

- $P(x)$ is of degree 5.
- 3 is a zero of $P(x)$ with some multiplicity m.

Additional properties may be specified or inferred, such as whether there are other roots, their multiplicities, or particular values of the polynomial at certain points.

Basic Constraints and Considerations

- The degree of the polynomial is 5.
- The total sum of multiplicities of the roots must equal 5.
- The root at 3 has multiplicity m, where $1 \leq m \leq 5$.
- The remaining roots (if any) must have multiplicities summing to $5 - m$.

Step-by-Step Construction Approach

Step 1: Decide on the multiplicity of 3

The multiplicity m can vary:

- $m = 1$: simple root at 3.
- $m = 2$: double root.
- $m = 3$: triple root.
- $m = 4$: quadruple root.
- $m = 5$: root at 3 with multiplicity 5 (a single root).

Each choice influences the shape and structure of the polynomial differently.

Step 2: Write the polynomial in factored form

The general form becomes:

$$
P(x) = a (x - 3)^m \times Q(x)
$$

where:

- $a \neq 0$ is a leading coefficient.
- $Q(x)$ is a polynomial of degree $5 - m$, representing the remaining roots.

Step 3: Determine the remaining roots and their multiplicities

Depending on the scenario, the remaining roots could be real or complex, distinct or repeated. For simplicity, consider the cases with real roots:

- Case A: The remaining roots are distinct and different from 3.
- Case B: The remaining roots coincide with other roots, possibly of multiplicity.

Practical Examples: Constructing Explicit Polynomials

Let's illustrate with specific examples for each possible multiplicity at 3.

Case 1: 3 as a Zero of Multiplicity 1

The simplest case:

$$
P(x) = a (x - 3) \times (x - r_2) \times (x - r_3) \times (x - r_4) \times (x - r_5)
$$

where r_2, r_3, r_4, r_5 are distinct roots, and a is any non-zero constant.

Example:

Choose $a=1$, roots at $3, 2, 4, 0, -1$:

$$
P(x) = (x - 3)(x - 2)(x - 4)(x)(x + 1)
$$

Expanding:

- First, multiply pairs:

$$
(x - 3)(x - 2) = x^2 - 5x + 6
$$
$$
(x - 4)(x)(x + 1) = (x - 4) \times x \times (x + 1)
$$

Alternatively, multiply all at once for the explicit polynomial form:

$$
P(x) = (x - 3)(x - 2)(x - 4)(x)(x + 1)
$$

which is a fifth-degree polynomial with the desired properties.

Case 2: 3 as a Zero of Multiplicity 2

In this case:

$$
P(x) = a (x - 3)^2 (x - r_3) (x - r_4) (x - r_5)
$$

Suppose the other roots are distinct, say, $0, 1, -2$.

Explicit example:

$$
P(x) = (x - 3)^2 (x)(x - 1)(x + 2)
$$

Expanding:

- Expand $(x - 3)^2 = x^2 - 6x + 9$.
- Multiply $(x^2 - 6x + 9)$ by x:

$$
x^3 - 6x^2 + 9x
$$

- Multiply the result by $(x - 1)$:

$$
(x^3 - 6x^2 + 9x)(x - 1) = x^4 - 7x^3 + 15x^2 - 9x
$$

- Multiply by $(x + 2)$:

$$
(x^4 - 7x^3 + 15x^2 - 9x)(x + 2)
$$

which, upon expansion, yields a fifth-degree polynomial.

Case 3: 3 as a Zero of Multiplicity 3

The polynomial takes the form:

$$
P(x) = a (x - 3)^3 (x - r_4) (x - r_5)

\]

Choose, for example, roots at 0 and -1:

$$
P(x) = (x - 3)^3 (x)(x + 1)
$$

This polynomial has a triple root at 3, with two other roots at 0 and -1.

Case 4: 3 as a Zero of Multiplicity 4

Now,

$$
P(x) = a (x - 3)^4 (x - r_5)
$$

Suppose the remaining root is at 2:

$$
P(x) = (x - 3)^4 (x - 2)
$$

This configuration results in a polynomial with a quadruple root at 3 and a simple root at 2.

Case 5: 3 as a Zero of Multiplicity 5

This is a degenerate case where the polynomial is:

$$
P(x) = a (x - 3)^5
$$

a monic polynomial with all roots at 3, multiplicity 5.

Additional Considerations: Choosing Coefficients and Roots

The above examples illustrate the construction process. Selecting the leading coefficient a allows for scaling, often chosen as 1 for simplicity unless specific constraints exist.

Furthermore, choosing roots other than 3 affects the polynomial's shape and symmetry. Real roots lead to polynomial graphs crossing the x-axis at those points, while complex roots contribute to the polynomial's coefficients but do not manifest as x-intercepts.

Theoretical Implications and Applications

Constructing such polynomials isn't merely an academic exercise. It has practical implications:

- Root multiplicity analysis: Understanding how multiplicities influence the polynomial's behavior near roots—important in numerical methods and root-finding algorithms.
- Polynomial interpolation: Designing functions with prescribed roots and multiplicities for modeling or approximation.
- Algebraic structure exploration: Investigating how roots

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